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While studying the work of Möbius, Prof. Zmodtwo discovers the importance of signs.

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This journal is devoted to cultivating mathematical reasoning and problem-solving skills and preparing students to face the challenges of the high-technology era.

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Cover Page: This picture was created for *π in the Sky* by Czech artist Gabriela Novakova, to illustrate an original idea of George Peschke. To discover the meaning of the scene depicted, see George Peschke's article on the back cover (page 32) of this issue.

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Math Opinion

This column is an open forum. We welcome opinions on all mathematical issues: research, education, and communication.

Back to Basics

A Bridge Too Far

by Scott Carlson [†]

> *"A widespread misconception about mathematics is that it is completely hierarchical—first arithmetic, then algebra, then calculus, then more abstraction."*
> Allen Paulos

Recently, it has become fashionable to criticize current mathematics curricula, and especially the use of calculators in primary and secondary schools. Generally, the criticisms are rooted in the misconception Paulos details above. Often, they are expressed in statements such as: "These kids don't know their times tables," or "Our students are unable to work with fractions." To many minds, these are the cardinal sins of mathematics. Without the ability to recite multiplication facts, it is thought, a person is condemned never to understand any other mathematics, because multiplication is the key to mathematical nirvana. Generally, the perceived villain is the calculator, thus the conclusion: if we remove the calculators from the classroom, all will be right with mathematics.

About a year ago, I was buying a stamp at the local post office. The 50-ish woman helping me spoke at length about the problems with math education today. It boiled down to the fact that kids do not know how to multiply, and are too dependent on calculators. Next, I expected a story about her trips to school three miles through two feet of snow, uphill both ways, all the while dragging her little brother. She then proceeded to dig out her calculator to find the GST on my 56 cent stamp. This is obviously ironic, but the real irony is a little more subtle. I am a product of the new, calculator-savvy generation, yet I could mentally calculate the GST before she could find her calculator. The new curriculum is certainly not responsible for her calculator dependence. Perhaps much of the criticism of calculator use is just a sublimation of nostalgic yearning for those happy days of yesteryear.

That is not to say basic skills and concepts are superfluous. The error is in thinking that no interesting or worthwhile mathematics is possible without first spending years reciting basic facts. I recently encountered an engineer who does not know her multiplication facts. Is this admirable? Clearly not. Is she proud of this? Again, no. But on the other hand, higher mathematics is obviously possible even with gaps in background knowledge. The reality is that not every person in this generation, nor any preceding generation, is flawless. This is not the fault of the calculator. The intelligent and appropriate use of calculators does not create these gaps, but mitigates them. The calculator, properly used, acts as a scaffolding to enable a person to get beyond minor gaps in background to higher mathematical concepts.

There are other advantages to the use of calculators and other technology in mathematics instruction. The graphing calculator helps make explicit the connections and differences between algebraic and graphic representations. It also helps highlight the advantages and disadvantages of each, depending on context. When calculators are used properly, students learn that graphs are a good way to see the global behaviour of a function, but that algebraic methods and representations are best for deciding what happens at any given point. In other words, students learn by experience that graphs yield approximations, yet algebraic methods are exact. Certainly, students have been told this before, but the calculator allows them to experience it. Though we may not like it, the reality is that experience is usually the best teacher.

As an example, for a given quadratic function, students can compare the value of an extremum obtained analytically to one obtained graphically. In many cases, there is slight disagreement. This may seem problematic at first, but in reality it offers an opportunity to focus on the difference in representations, and on the importance of choosing methods that achieve the goal in the most accurate and desirable way. When technology is used effectively, the student sees its shortcomings, and understands that the calculator is not an infallible black box. In this case, the teacher can capitalize on the desire for a quick and painless solution, and demonstrate that quick and painless often sacrifices accuracy. In particular, using the calculator does not mean students no longer need to learn to complete the square; instead, they see firsthand that completing the square is the more accurate approach. At the same time, they acquire a device that helps them incorporate their intuition and check their answers.

In his book *The Math Gene*, Keith Devlin defines mathematics as "the science of patterns." Devlin is neither the first nor the last to define mathematics this way. In fact, most practicing mathematicians would likely give a similar definition. Pure mathematicians, applied mathematicians, and statisticians find and prove theorems based on the patterns they find. Much of the beauty of math is in the generalization of results; that is, the application of patterns to broader and broader places. Used intelligently, calculators help students find patterns for themselves. Those with a utilitarian bent may say that it is more efficient to tell the students what the patterns are. However, if there is one thing that we have learned from research and experience, it is that telling is not teaching. Or, as the old proverb goes:

I hear I forget,
I see I remember,
I do I understand.

While it may be more efficient to tell students the rules and expect them to memorize, learning is not an efficient process. Much more learning occurs when students experiment and find patterns for themselves. Before the advent of technology, this was an impractical position, but this is no longer true. We can now design effective investigations so that students can uncover mathematical principles and patterns for themselves.

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For example, rather than just telling students that the graph of distance versus time for a falling object is a parabola, handheld technology enables students to experiment with falling objects, collect data firsthand, and see the connection between the motion and the graph themselves. The students can repeat the experiment as many times as they need to, with numerous objects. This also gives the teacher a natural reason to talk about quadratics, properties of parabolas, and the relevant ideas and definitions. The calculator does not replace the need for students to know about quadratics, but it makes the knowledge more accessible. In my own experience, a carefully designed activity can communicate the important properties and definitions of parabolas to students in the same amount of time as a traditional lecture. However, days later, more students will remember the terms, and at least one instance where a quadratic is important. Calculators don't replace thinking; they enable it, when intelligently used.

The most obvious argument for graphing technology is that it enables visualization. The best way to have students develop the ability to visualize parabolas, for instance, is establishing familiarity by seeing many of them, with varying orientations and positions. The easiest way to do this is to let students graph as many as necessary with some sort of fast graphing tool, such as a calculator or computer. The idea is not to excuse students from knowing what a parabola looks like, how it can be transformed, or how its equation determines its graph—rather, it is to enable students to find the underlying patterns and principles first-hand. Later, this will allow students to visualize parabolas more easily when using them in other areas of mathematics.

Similarly, if the task is finding volumes of solids of revolution, the logical place to begin is with a sketch of the function that determines the solid. With a graphing calculator, students can quickly obtain a graph, and then visualize the solid of revolution. In this way, the students are able to spend their time and energy on the appropriate calculus concepts. The calculator is not thinking for the students, who are still required to choose the appropriate method and perform the required integration.

Educators and non-educators alike often dismiss new practices in education as fads, or worse. Many have criticized the use of calculators using such logic. Interestingly, rarely do the critics give any objective evidence to support their position. Basing a conclusion on small, non-random samples is clearly a bad practice, and mathematicians in particular should not make this error. In 1999, Penelope Dunham conducted a review of the research on calculator use in mathematics education. After reviewing literally dozens of studies conducted since graphing calculators were introduced in 1986, she found several trends:

- *"Students who use graphing calculators display better understanding of function and graph concepts, improved problem solving, and higher scores on achievement tests for algebra and calculus skills.*
- *Students who learn paper-and-pencil skills in conjunction with technology-based instruction and are tested without calculators perform as well or better than students who do not use technology in instruction.*
- *Those (teachers) who support mastery first often view mathematics as computation rather than a process for patterning, reasoning, and problem solving."*

She found, not surprisingly, that calculator use did not eliminate student error, and even that a class of calculator-induced errors existed. Her conclusion, though, is unequivocal: "Handheld technology can and should play an important role in mathematics instruction."

Another source of objective data is the recent (1999) TIMSS-R study. This study compares achievement in math and science for large, random samples of students from 39 countries. For Canadians, and especially Albertans, there is plenty of good news in the results. Canadian grade eight students scored significantly higher than the international average on the math exam. In fact, only six countries had averages significantly higher than Canadian students. Alberta students did even better. Not only is this excellent news, it discredits the argument that calculator use inhibits the development of basic skills—the use of calculators was forbidden on the exam.

To go further, in several of the countries that scored higher than Canada in the study, calculators are used as much as, or more than, in Canada. For example, in Belgium, calculator use is compulsory after grade nine. In the Netherlands, calculators are compulsory for national exams and for grades 11 and 12. In Hong Kong, calculator use is unrestricted after grade seven; in Japan, it's after grade five. The calculator has clearly contributed to the success of mathematics students the world over.

Calculators are not the origin of society's innumeracy. Most of the parents of the students I have taught learned mathematics before the advent of calculators in schools, and certainly before graphing calculators were conceived. Commonly, they confide that they succeeded in school mathematics until they were introduced to fractions. This perplexes me. If calculators are to shoulder the blame for society's lack of fluency with rational numbers, how did they perpetrate this crime in advance? Perhaps the technology is more potent than we supposed.

Calculators are certainly not a panacea for society's innumeracy. Undoubtedly, students and teachers have used calculators improperly. The solution is not, as some advocate, to discard or prohibit the technology. Many university faculty are notorious for their misuse of chalkboards, but no one suggests that the solution is to confiscate the chalk. Calculators should not be used to replace logic, thinking, or algebra. No reasonable person advocates such replacement, nor does our public school curriculum.

Since the technology is likely to become more pervasive, the mathematics community must encourage uses of technology that develop logical, appropriate mathematical thinking. Certainly this means a change in emphasis in some math courses, but this is not equivalent to lowering our standards. We will likely be required to recognize mathematically correct and appropriate work in formats that are new to us, but this has been required of others before us. Even in mathematics, change is inevitable. We may not all like the Emperor's new wardrobe, but it still covers the essentials.

References:

[1] Paulos, John Allen. Beyond Numeracy. Alfred A. Knopf, New York, 1991.

[2] Devlin, Keith. The Math Gene. Basic Books, 2000.

[3] Dunham, Penelope H. Hand-Held Calculators in Mathematics Education: A Research Perspective, in Hand-Held Technology in Mathematics and Science Education: A Collection of Papers, E. D. Laughbaum, Editor. Ohio State University, 2000.

[4] Mullis *et al.* TIMMS 1999 International Mathematics Report. Boston College and the International Association for the Evaluation of Educational Achievement, 2000.

VIEWPOINTS

Mathematics of the Past

by

Garry Kasparov

Since my early childhood, I have been inspired and excited by ancient and medieval history. I also have a good memory, which allows me to remember historical events, dates, names, and related details. So, after reading many history books, I analysed and compared the information and, little by little, I began to feel that there was something wrong with the dates of antiquity. There were too many discrepancies and contradictions that could not be explained within the framework of traditional chronology. For example, let's examine what we know of ancient Rome.

Edward Gibbon (1737-1794)

The monumental work *The Decline and Fall of the Roman Empire*, written by English historian and scholar Edward Gibbon (1737-1794), is a great source of detailed information on the history of the Roman Empire. Before commenting on this book, let me remark that I cannot imagine how—with their vast territories—the Romans did not use geographical maps, how they conducted trade without a banking system, and how the Roman army, on which the Empire rested, was unable to improve its weapons and military tactics during nine centuries of wars.

With the use of simple mathematics, it is possible to discover in ancient history several such dramatic contradictions, which historians don't seem to consider. Let us analyze some numbers. E. Gibbon gives a very precise description of a Roman legion, which *" ...was divided into 10 cohorts... The first cohort,... was formed of 1 105 soldiers... The remaining 9 cohorts consisted each of 555 soldiers,... The whole body of legionary infantry amounted to 6 100 men."*[1] He also writes, *"The cavalry, without which the force of the legion would have remained imperfect, was divided into 10 troops or squadrons; the first, as the companion of the first cohort, consisted of a 132 men; while each of the other 9 amounted only to 66. The entire establishment formed a regiment... of 726 horses, naturally connected with its respected legion... "*[2] Finally, he gives an exact estimate of a Roman legion: *"We may compute, however, that the legion, which was itself a body of 6 831 Romans, might, with its attendant auxiliaries, amount to about 12 500 men. The peace establishment of Hadrian and his successors was composed of no less than 30 of these formidable brigades; and most probably formed a standing force of 375 000."*[3] This enormous military force of 375 000 men, maintained during a time of peace, was larger than the Napoleonic army in the 1800s.[4] Let me point out that according to the Encyclopædia Britannica,[5] *"Battles on the Continent in the mid-18th century typically involved armies of about 60 000 to 70 000 troops."* Of course, an army needed weapons, equipment, supplies, etc. Again, E. Gibbon gives us a lot of details:[6] *"Besides their arms, which the legionaries scarcely considered as an encumbrance, they were laden with their kitchen furniture, the instruments of fortifications, and the provisions of many days. Under this weight, which would oppress the delicacy of a modern soldier,*[7] *they were trained by a regular step to advance, in about six hours, nearly twenty miles. On the appearance of an enemy, they threw aside their baggage, and by easy and rapid evolutions converted the column of march into an order of battle."* This description of the physical fitness of an average Roman soldier is extraordinary. It brings us to the very strange conclusion that, at some point, the human race retrogressed in its ability to cope with physical problems. Is it possible that there was a gradual decline of the human race, with hundreds of thousands of Schwarzenegger-like athletes of Roman times evolving into medieval knights with relatively weak bodies (like today's teenage boys), whose little suits of armor are today proudly displayed in museums? Is there a reasonable biological or genetic explanation to this dramatic change affecting the human race over such a short period of time?

In order to supply such an army with weapons, a whole industry would have been needed. In his work, E. Gibbon explicitly mentions iron (or even steel) weapons: "*Besides a lighter spear, the legionary soldier grasped in his right hand the formidable pilum..., whose utmost length was about six feet, and which was terminated by a massy triangular point of steel of eighteen inches."*[8] In another place, he indicates *"The use of lances and of iron maces... "*[9] It is believed that the extraction of iron from ores was very common in the Roman Empire.[10] However, to smelt pure iron, a temperature of 1 539°C is required, which couldn't be achieved by burning wood or coal without the blowing or the blast furnaces invented more than a 1000 years later.[11] Even in the 15th century, the iron produced was of quite poor quality because large amounts of carbon had to be absorbed to lower the melting temperature to 1 150°C. There is also the question of sufficient resources—the blast furnaces used in the mid-16th century required large amounts of wood to produce charcoal, an expensive and unclean process that led to the eventual deforestation of Europe. How could ancient Rome have sustained a production of quality iron on the scale necessary to supply thousands of tonnes of arms and equipment to its vast army?

Just by estimating the size of the army, we can conclude that the population of the Eastern and Western Roman Empire in the second century AD was at least 20 million people, but it could have been as high as 40 or even 50 million. According to E. Gibbon, *"Ancient Italy... contained eleven hundred and ninety seven cities."*[12] The city of Rome had more

[1] See [1], page 30.

[2] See [1], p. 32.

[3] See [1], p. 35.

[4] After 1800, Napoleon routinely maneuvered armies of 250 000. See the Encyclopædia Britannica.

[5] Encyclopædia Britannica online at `http://www.britannica.com/`

[6] See [1], p. 35.

[7] E. Gibbon wrote these words in the years 1776–88, before the French Revolution and the Napoleonic wars.

[8] See [1], p. 31.

[9] See [1], p. 33.

[10] See Encyclopædia Britannica.

[11] See [7], where the presented facts prove that real metallurgy started in the 16th century. Coal was discovered in England only in the 11th century.

[12] See [1], page 71.

than a half-million inhabitants, and there were other great cities in the Empire. All of these cities were connected by a network of paved public highways, their combined lengths totalling more than 4000 miles![13] This could only be possible in a technologically advanced society. According to J.C. Russell,[14] in the 4th century, the population of the Western Roman Empire was 22 million (including 750 000 people in England and five million in France), while the population of the Eastern Roman Empire was 34 million.

It is not hard to determine that there is a serious problem with these numbers. In England, a population of four million in the 15th century grew to 62 million in the 20th century. Similarly, in France, a population of about 20 million in the 17th century (during the reign of Louis XIV), grew to 60 million in the 20th century—and this growth occurred despite losses due to several atrocious wars. We know from historical records that during the Napoleonic wars alone, about three million people perished, most of them young men. But there was also the French Revolution, the wars of the 18th century in which France suffered heavy losses, and the slaughter of World War I. By assuming a constant population growth rate, it is easy to estimate that the population of England doubled every 120 years, while the population of France doubled every 190 years.

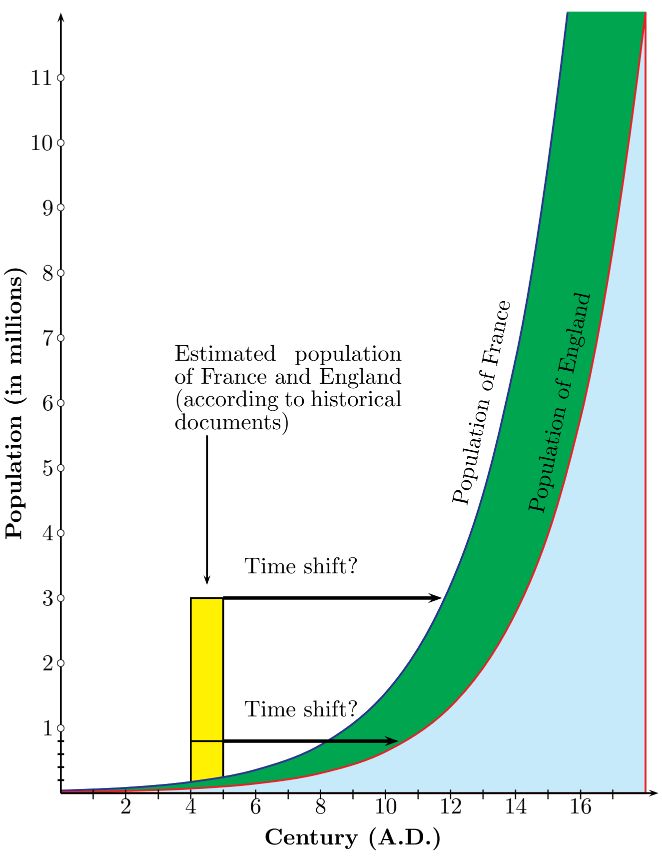

Figure 1

Graphs showing the hypothetical growth of these two functions are provided in Figure 1. According to this model, in the 4th and 5th centuries, at the breakdown of the Roman Empire, the (hypothetical) population of England would have been 10 000 to 15 000, while the population of France would have been 170 000 to 250 000. However, according to estimates based on historical documents, these numbers should be in the millions.

It seems that starting with the 5th century, there were periods during which the population of Europe stagnated or decreased. Attempts at logical explanations, such as poor hygiene, epidemics, and short lifespan, can hardly withstand criticism. In fact, from the 5th century until the 18th century, there was no significant improvement in sanitary conditions in Western Europe, there were many epidemics, and hygiene was poor. Also, the introduction of firearms in the 15th century resulted in more war casualties. According to UNESCO demographic resources, an increase of 0.2 per cent per annum is required to assure the sustainable growth of a human population, while an increase of 0.02 per cent per annum is described as a demographical disaster. There is no evidence that such a disaster has ever happened to the human race. Therefore, there is no reason to assume that the growth rate in ancient times differed significantly from the growth rate in later epochs.

These discrepancies lead me to suspect that there is a gap between the historical dates attributed to the Roman Empire and those suggested by the above computations. But there are more inconsistencies in the historical record of humankind. As I have already noted, there are similar gaps of several centuries in technological and scientific development. Notice that knowledge and technology traditionally associated with the ancient world presumably disappears during the Dark Ages, only to resurface in the 15th century during the early Renaissance. The history of mathematics provides one such example. By chronologically and logically ordering major mathematical achievements, beginning with arithmetic and Greek geometry and finishing with the invention of calculus by I. Newton (1643–1727) and G.W. Leibnitz (1646–1716), we see a thousand-year gap separating antiquity from the new era. Is this only a coincidence? But what about astronomy, chemistry (alchemy), medicine, biology, and physics? There are too many inconsistencies and unexplained riddles in ancient history. Today, we are unable to build simple objects made in ancient times in the way they were originally created[15]—this in a time when technology has produced the space shuttle and science is on the brink of cloning the human body! It is preposterous to blame all of the lost secrets of the past on the fire that destroyed the Library of Alexandria, as some have suggested.

It is unfortunate that each time a paradox of history unfolds, we are left without satisfactory answers and are persuaded to believe that we have lost the ancient knowledge. Instead of disregarding the facts that disagree with the traditional interpretation, we should accept them and put the theory under rigorous scientific scrutiny. Explanations of these paradoxes and contradictions should not be left only to historians. These are scientific and multidisciplinary problems and, in my opinion, history—as a single natural science—is unable on its own to solve them.

I think that the chronology of technological and scientific development should be carefully investigated. The too-numerous claims of technological wonders in antiquity turn history into science fiction (e.g., the production of monolithic stone blocks in Egypt, the precise astronomical calculations obtained without mechanical clocks, the glass objects and mirrors made 5000 years ago,[16] and so on). It is un-

[13] See [1], page 74.

[14] See [6].

[15] For example, try to build a working wheel according to ancient diagrams, but do it without using iron or iron tools.

[16] Making glass, in technical terms, is a secondary product of black metallurgy requiring a temperature of 1 280°C.

fortunate that historians reject scientific incursion into their domain. For instance, the most reasonable explanation of Egyptian pyramid-building technology, presented by French chemist Joseph Davidovits[17] (the creator of the geopolymer technology), was rejected by Egyptologists, who refused to provide him with samples of pyramid material.

About five years ago, I came across several books written by two mathematicians from Moscow State University: academician A.T. Fomenko and G.V. Nosovskij. The books described the work of a group of professional mathematicians, led by Fomenko, who had considered the issues of ancient and medieval chronology for more than 20 years, with fascinating results. Using modern mathematical and statistical methods,[18] as well as precise astronomical computations,[19] they arrived at the conclusion that ancient history was artificially extended by more than 1000 years. For reasons beyond my understanding, historians are still ignoring their work.

Greek and Roman Counting System

Modern	Greek	Roman	Modern	Greek	Roman
1	α	I	25	$\kappa\varepsilon$	XXV
2	β	II	50	ν	L
3	γ	III	70	o	LXX
4	δ	IV	80	π	LXXX
5	ε	V	100	ρ	C
6	ς	VI	200	σ	CC
7	ζ	VII	500	ϕ	D
8	η	VIII	800	ω	DCCC
9	θ	IX	1000	${}^{\iota}\alpha$	M
10	ι	X	10000	M^{α}	$\overline{\text{X}}$
20	κ	XX	20 000	M^{β}	$\overline{\text{XX}}$
24	$\kappa\delta$	XXIV	100 000	M^{ι}	$\overline{\text{C}}$

Table 1

But let us return to mathematics and to ancient Rome. The Roman numeral system discouraged serious calculations. How could the ancient Romans build elaborate structures such as temples, bridges, and aqueducts without precise and elaborate calculations? The most important deficiency of Roman numerals is that they are completely unsuitable even for performing a simple operation like addition, not to mention multiplication, which presents substantial difficulties (see Table 1).[20] In early European universities, algorithms for multiplication and division using Roman numerals were doctoral research topics. It is absolutely impossible to use clumsy Roman numbers in multi-stage calculations. The Roman system had no numeral "zero." Even the simplest decimal operations with numbers cannot be expressed in Roman numerals.

Just try to add Roman numerals:[21]

MCDXXV
+
MCMLXV,

or multiply:[22]

DCLIII
×
CXCIX.

Try to write a multiplication table in Roman numerals. What about fractions and operations with fractions?

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Despite all of these deficiencies, Roman numerals supposedly remained the predominant representation of numbers in European culture until the 14th century. How did the ancient Romans succeed in their calculations, including complicated astronomical computations? It is believed that in the 3rd century, the Greek mathematician Diophantus was able to find positive and rational solutions to the following system of equations, called *Diophantic* today:

$$x_1^3 + x_2 = y^3,$$
$$x_1 + x_2 = y.$$

According to historians, at the time of Diophantus, only one symbol was used for an unknown, a symbol for "plus" did not exist; neither was there a symbol for "zero." How could Diophantic equations be solved using Greek letters or Roman numerals (see Table 1)? Can these solutions be reproduced? Are we dealing here with another secret of ancient history that we are not supposed to question? Let us point out that even Leonardo da Vinci, at the beginning of the 16th century, had troubles with fractional powers.[23] It is also interesting that in all of da Vinci's works, there is no trace of "zero" and that he was using 22/7 as an approximation of π—probably it was the best approximation of π available at that time.[24]

It is also interesting to look at the invention of the logarithm. The logarithm of a number x (to the base 10) expresses simply the number of digits in the decimal representation of x, so it is clearly connected to the idea of the positional numbering system. Obviously, Roman numerals could not have led to the invention of logarithms.

Knowledge of our history timeline is important, and not only for historians. If indeed the dates of antiquity are incorrect, there could be profound implications for our beliefs

[17] See [3].
[18] See [4].
[19] See [5].
[20] Even in 1768, in the first edition of Encyclopædia Britannica, there were some variations in the use of the Roman numerals. For example, the symbol IIII was sometimes used instead of IV for the number four.
[21] Answer: MMMCCCXC. You can check your work with this online Roman numeral calculator: http://www.naturalmath.com/tool2.html.
[22] Answer: $\overline{\text{CXX}}\text{M}\overline{\text{X}}\text{CMXLVII}$.
[23] Da Vinci made a mistake in his computations of the area of a cross-section of a cube—he wasn't able to express his result, which contained the fractional power 3/2. See [8], F., p. 59.
[24] See [9], p. 1.

about the past, and also for science. Historical knowledge is important to better understand our present situation and the changes that take place around us. Important issues such as global warming and environmental changes depend on available historical data. Astronomical records could have a completely different meaning if the described events took place at times other than those provided by traditional chronology. I trust that the younger generation will have no fear of "untouchable" historical dogma and will use contemporary knowledge to challenge questionable theories. For sure, it is an exciting opportunity to reverse the subordinate role science plays to history, and to create completely new areas of scientific research.

References:

[1] E. Gibbon. The Decline and Fall of the Roman Empire. Peter Fenelon Collier & Son, vol. 1, New York, 1899. This book is also available online at:
http://www.ccel.org/g/gibbon/decline/.

[2] I. Davidenko and Y. Kesler. Book of Civilization, ABRIDGEMENT (with preface by Garry Kasparov). EkoPress-2000, Moscow, 2001, translated to English by Ludmila Zoikina.

[3] J. Davidovits and M. Morris. The Pyramids: An Enigma Solved. New York: Hippocrene Books, 1988 (4th printing). Later by Dorset Press, New York, 1989, 1990.

[4] A. T. Fomenko. Empirico-Statistical Analysis of Narrative Material and its Applications to Historical Dating. Volume 1: The Development of the Statistical Tools, and Volume 2: The Analysis of Ancient and Medieval Records. Kluwer Academic Publishers, 1994, The Netherlands.

[5] A. T. Fomenko , V.V. Kalashnikov and G.V. Nosovskij. Geometrical and Statistical Methods of Analysis of Star Configurations: Dating Ptolemy's Almagest. CRC Press, 1993, USA.

[6] J. C. Russell. Late Ancient and Medieval Population. American Philosophical Society. 152 p., (Transactions of the American Philosophical Society 48 pt. 3), Philadelphia, 1958.

[7] J.E. Dayton. Minerals, Metals, Glazing and Man. Harrap, London, 1978. ISBN: 0245528075.

[8] The Notebooks of Leonardo da Vinci, 2nd ed., 2 vol. (1955, reissued 1977); and Jean Paul Richter (compiler and ed.). Original kept at Institut de France, Paris.

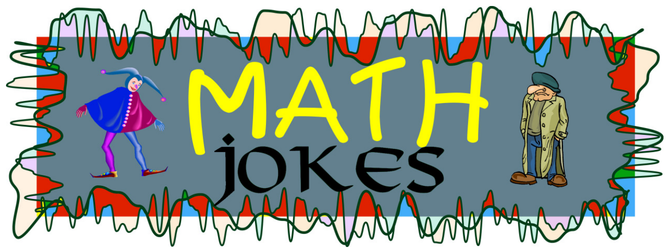
[9] Leonardo da Vinci. Codex Atlanticus. Kept in Biblioteca Ambrosiana in Milan, Italy.

Garry Kasparov has been the chess world champion since 1985, when he won the title at the age of 22. In 1997, during a historical chess challenge that made headlines all over the world, he defeated IBM's Deep Blue supercomputer. There are many web sides devoted to Garry, but we recommend:

http://www.kasparovchess.com/.

A biography can be found at

http://www.chennaiweb.com/sp/chess/bio/garyk/.



The math professor's six-year-old son knocks at the door of his father's study.

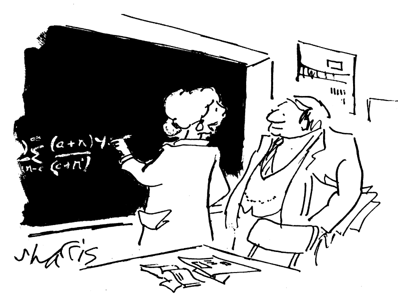
"Daddy," he says. "I need help with a math problem I couldn't do at school."

"Sure," the father says and smiles. "Just tell me what's bothering you."

"Well, it's a really hard problem: *There are four ducks swimming in a pond when two more ducks come and join them. How many ducks are now swimming in the pond?"*

The professor stares at his son in disbelief. "You couldn't do that?! All you need to know is that $4 + 2 = 6$!"

"Do you think, I'm stupid?! Of course I know that $4 + 2 = 6$. But what does this have to do with ducks!?"



"The beauty of math, of course, is that we don't even need an ethicist"

A visitor to the Royal Tyrell museum in Alberta asks a museum employee:

"How old is the skeleton of that T-Rex?"

"Precisely 60 million and three years, two months, and 12 days."

"How can you know that with such precision?"

"That's easy. When I started working here, a sign said that the skeleton was 60 million years old. And that was three years, two months, and 12 days ago..."

"What is π?"

A mathematician: "π is the ratio of the circumference of a circle to its diameter."

A computer programmer: "π is 3.141592653589 in double precision."

A physicist: "π is 3.14159 plus or minus 0.000005."

An engineer: "π is about 22/7."

A nutritionist: "Pie is a healthy and delicious dessert!"

Q: How do you make one burn?

A: Differentiate a log fire!

COUNTERPOINTS

Mathematics: A Tool for Questioning

by
Nassif Ghoussoub and
Klaus Hoechsmann

Garry Kasparov

In the preceding article, the man who defeated the world's best chess champions and IBM's formidable "Deep Blue" computer has done π *in the Sky* an invaluable favour: by using mathematics to examine the world around him, past and present, he is greatly contributing to our mission of raising mathematical awareness, stimulating analytical thinking, and encouraging critical questioning of widely-held beliefs. Mathematics (Greek for "learning") should be cultivated as a tool for systematic questioning, our primary defense against mumbo-jumbo and demagoguery.

Kasparov's message is simple: "Do not accept authority unquestioned—look for yourself." The first authority he questions is that of Edward Gibbon, whose Decline and Fall of the Roman Empire is a monument not only to history but also to English prose. But wherever numbers are involved, you can jump in and at least check the arithmetic. Adding up the cohorts of infantry and cavalry is probably not done by most readers of Gibbon, but it is easy (9 times 555 equals 5 times 999, etc.) and fun. In the end you come up with 6826 (Gibbon has five more, perhaps officers) and have to multiply that by 30. No calculators are allowed: your number lies just short of half-way between 6667 and 7000, hence the total will come to about 205 000. To get from there to his "standing force of 375 000," Gibbon has to add 170 000 "attendant auxiliaries," almost one per soldier. Why so many? Did the Romans never have government cutbacks? With one auxiliary for every five soldiers (is that reasonable?) the total force would be less than 250 000, the number given for Napoleon.

To play around with these numbers some more, you can try to visualize how big a square one-quarter million men would occupy if each man occupies one square meter. Or you can distribute them on the 4000 miles of paved highway the Romans had (according to Gibbon). How far apart would they stand? If the Empire had 50 million inhabitants, that size of army would comprise one percent of the male population. If their life expectancy was 50 years, how long would their military service have to be to arrive at that number? As you can see, historical writings can provide an almost endless source of such exercises. Why should arithmetic and history always be taught separately?

Thomas R. Malthus

After wondering about the feasibility of some of the Roman marvels reported by Gibbon (for instance, the steel required to equip each legionnaire with a "pilum"), Kasparov's curiosity turns to the work of another famous Englishman, whom he however does not name. In political circles, that name invariably unleashes heated and bitter debates, because its owner wrote in 1798 that "population increases in a geometric ratio, while the means of subsistence increases in an arithmetic ratio." We are, of course, talking about Thomas R. Malthus. What does he mean? Population grows by perpetual multiplication (exponentially), while food production grows only by repeated addition (linearly); in other words, humanity is doomed!

Malthus does not leave it at these vague pronouncements, but says in his *Essay on the Principles of Population* (Chapter 2) that "population, when unchecked, goes on doubling itself every twenty-five years," after citing "the United States of America, where the means of subsistence have been more ample, the manners of the people more pure..." The phrase "when unchecked" throws a big spanner into the works: we are now at 200 years (eight doubling periods after Malthus), but have not doubled the world population of his time (about one billion) eight times; otherwise we'd now be at 256 billion instead of "only" six. Going backward in time, where Malthus would reduce the population by 50 percent every 25 years, similar nonsense would result. In working with doubling or halving, it is convenient to remember that the 10th power of 2 is 1024. Going back in time 250 years (10 Malthusian doubling times), he would go from one billion to one million—two more such large steps (750 years in total), and he would arrive at Adam. That's why these calculations need the condition "unchecked."

There are situations where this condition is almost satisfied. If you take a culture of bacteria in plenty of nutrient solution—they have no wars and do not practise birth control—you can observe (almost) pure exponential growth. And in radio-active decay—because atoms don't make choices—you can see it in reverse: every so many years (always the same number, called the "half-life"), the remaining "population" of radio-active atoms is halved. For radio-active carbon, the half-life is about 5700 years. When a plant or animal ceases to take part in the great cycle of life, its carbon content remains static, and the radio-active part of it decays with that fixed half-life. So if you find a piece of wood with only one-quarter the "typical" amount of radioactive carbon, you would presume that it has been dead for about 11 000 years.

But let us get back to human populations, where growth is apparently not "unchecked." It does not help, in the long run, to assume a greater doubling time: whatever length of step you choose, after 30 such steps back in time, you'll knock off nine zeroes, going from the present six billion to a mere six individuals—the Garden of Eden. In the medium run, you might observe something resembling exponential growth—but don't count on it. Look at the recent past: in 1800 we were one billion, in 1935 we were two billion, in 1975 we were four billion. The sad truth is that our doubling time seems to be shrinking. Pretty soon, it will be at the 25-year level assumed by Malthus—it looks as though the Old Man was not pessimistic enough.

Kasparov's inquisitiveness is not random but has a theme: exactly how long ago was it that the Romans had their Empire? At first glance, this question is surprising (don't we all know about those 2000 years?), but on second thought it is entirely legitimate. Anyone with a scientific bent of mind will put more trust in directly accessible data (e.g., the movement of stars) than in stories told by knights and monks—especially if these are vague and contradictory. According to people who study old manuscripts, medieval European record-keeping was a mess, and so it seems that some scrupulous revision is in order. The same scientific spirit that allows the question, however, compels us to question any answer—in this case, the one proposed by Fomenko's Moscow team. Since everyone seems to agree that time-keeping was fairly good from Caesar until about 400 AD and then again since Galileo (at least!), we have only about 1 200 possibly "sloppy" years to straighten out. If Islamic history, which is "modern" compared to most others, turns out to be as reliable as it looks, these uncertain years might shrink to a mere 200. For instance, the idea suggested in the article by Krawcewicz on page 12 of this issue, that "pagan" Egyptian frescoes could have been painted 600 years ago, would itself become rather questionable, if it were shown that Egypt was solidly Islamic at the time. That does not invalidate the author's study—it only shows that history is less certain than we sometimes think. Until the dust has settled, it is advisable not to pass judgment.

If the Roman Empire is really so far removed from us in time, why is it that Roman numerals were still in commercial use until the 14th century? Before we throw our own guess into the debate, let us look at the nature of these much maligned numerals. How could anyone calculate with them? Well, how can anyone compute "three hundred and seventy-six times two hundred and thirty-seven." You type these data into your pocket calculator and press the "×" button, that's how. You certainly would not fill page after page with number words. Neither did the Romans. They would load CCCLXXVI and CCXXXVII onto their counting board or abacus and manipulate the pebbles and beads until they had the result. We shall do such a multiplication, but first we'll look at addition and subtraction.

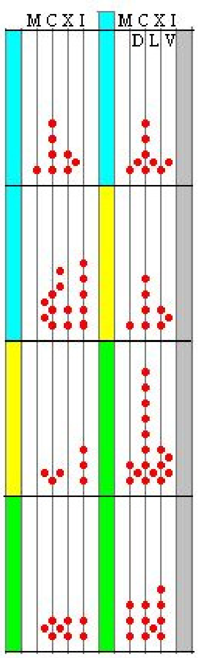


Figure 1

The counting board shown in Figure 1 is divided into two vertical strips; the left one is for subtraction and the right one is for addition. Let's do addition first. The number shown in the top-right field is MDCCCCLXV; the number immediately below is MCCCCXXV. To add them, we just pile everything together into the mess shown in the third field on the right. To make it readable, we have to reduce it—any five "beads" on a line are converted to one "button" in the space to the left of that line, and any two buttons in a space are converted to a single bead on the line immediately to the left. The answer is MMMCCCLXXXX, as shown in the bottom right field.

Note: we use the term "beads" to remind you of an abacus; our "buttons" would be found in the separate top compartment (called "heaven" by the Chinese) of the abacus. We are ignoring the medieval convention of writing IV, XL, CD instead of the longer but clearer IIII, XXXX, CCCC notation used by the ancients.

In the subtraction on the left strip, the first number MCCCCXXV must be expanded in order to have enough beads on each line and buttons in each space to allow the second number DCLIII, depicted in the third field, to be subtracted. The expansion, which is reduction in reverse, is shown in the second field from the top. It need not be done all at once, but can be performed as needed for subtraction. Answer: DCCLXXII.

The power and flexibility of the Roman numeral system is best demonstrated in how it handles multiplication: because of the numbers V, L, D, etc., you need not memorize any multiplication table beyond five. But five itself is just 10 halves, and halving is an easy operation. Doubling is another easy operation, and quadrupling is just doubling twice—so the hardest multiplier is three. If you do happen to know the 10-by-10 table, you can read every line together with its preceding space as a single decimal digit, and thus increase your speed.

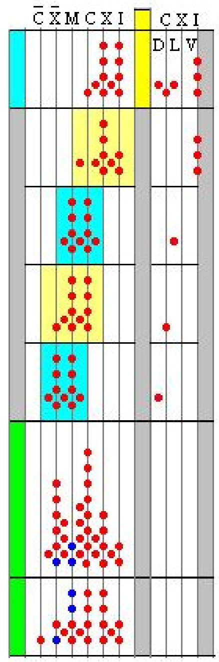


Figure 2

The multiplication shown in Figure 2 is CLXXXXVIIII times DCLIII. There are four partial products (in the blue and yellow fields) corresponding to the four digits of the multiplier: three, five (shifted), one (shifted twice), and five (shifted twice). As you pile all that into the first of the fields marked green, something special happens on the M-line: three sets of four. Since there is no space for that many, you turn them into a 12 (cf. blue beads) and carry on. After reducing this, you get $\overline{\text{CXXV}}$MMMMDCCCCXXXXXVII, as shown in the bottom field. If you find this too long, compare it to "one hundred twenty-nine thousand nine hundred and forty-seven."

A Roman wine merchant would have done this in his head: CLXXXXVIIII is one less than CC, so double DCLIII to MCCCVI, shift to $\overline{\text{CXXX}}$DC, and subtract DCLIII, and that'll be LIII short of $\overline{\text{CXXX}}$—factus est.

After all of this, you must be dying to see a division, and here it is: MMMMDCXXVIIII divided by XIII (the divisor is not entered in). It goes just as you expect. Since XIII takes up two lines, you look at the first two lines (plus spaces) of the number to be divided, and you see XXXXVI, which can accommodate three times XIII.

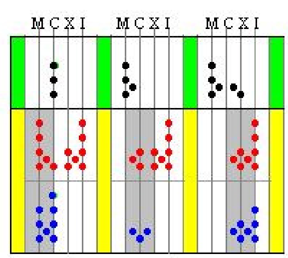


Figure 3

So you write a III on the line where your XXXXVI had its I. Then you subtract III times XIII and are left with VII, which is really DCC in disguise. Then you repeat the game, this time taking aim at what looks like LXXII—and so on, always wandering toward the smaller values on the right (see Figure 3).

To appreciate the ease and freedom of this simple gadget, you owe it to yourself to try one. For starters, why not take a chessboard and a supply of pennies? You can start your calculations on the right or on the left, change direction when you spot an opportunity for an easy move—as long as you keep track of where you are in the calculation, it cannot go wrong. You can add or subtract tokens to undo a lousy move—you never need an eraser.

The Indo-Arabic numeral system was supposedly introduced to Europe in the early 13th century with a book called Liber Abaci (book of the abacus) written by the widely travelled Leonardo da Pisa (alias Fibonacci), himself no mean mathematician. Present-day scholars say that it was known in the West much earlier—though still regarded as a Levantine curiosity—but that the 13th century introduction of paper from China, as a cheap medium for writing, made it the system of choice for all auditors and tax-collectors who wanted to see the details of every calculation.

The pen-on-paper computation with Indo-Arabic numerals—including the famous zero (originally a punctuation mark)—made it possible to check calculations for errors, but also penalized false starts and other trivial mistakes with ugly and confusing erasures. To avoid these, you had to follow certain very tight algorithms, which to this day make elementary arithmetic an incomprehensible and unpleasant discipline to many people. As Scott Carlson points out in the article preceding Kasparov's, the paper method makes little sense when a calculator is at hand—although mental arithmetic is something he evidently likes. To build the bridge between the two, how about re-introducing the counting board?

This ancient and user-friendly tool was still being used in Europe long after people had begun writing numbers in the more compact Indo-Arabic style. As late as 1550, a German textbook was published by one Adam Ries, in which the multiplication shown above would be written as 199 times 653 equals 129 947, but the intermediate steps would be left as unnamed patterns on the board. Even the Chinese and Japanese use this style to write input and output of their abacus work, and this would probably be the right way to bridge the gap between mental arithmetic and the calculator.

In conclusion: the counting board survived (at least) until the 16th century, and for a while (we guess) just carried the Roman numerals along with it. The fact that they are harder to falsify may also have helped.

The last major question raised by Kasparov concerns Diophantus of Alexandria. This Greek working in Roman times, considered the "father" of number theory, is indeed an enigma for anyone interested in chronology—the guesses about his dates range from 150 BC to 350 AD. If he lived that long ago, at a time when equations were allowed only one unknown (called the "arithm"), how could he have solved equations like "y cubed minus x cubed equals y minus x"? Here is what the Master himself says in Book IV, Problem 11 of his *Arithmetica*, according to the French translation by Paul Ver Eeke (1959), here rendered in English:

"To find two cubes having a difference equal to the difference of their sides. Suppose the sides to be 2 arithms and 3 arithms. Then the difference of the cubes with these sides is 19 cube arithms, and the difference of their sides is 1 arithm. Consequently, 1 arithm equals 19 cube arithms, and the arithm cannot be rational, because the ratio between these quantities is not like that of one square to another. We are thus led to look for cubes such that their difference is to the difference of their sides as one square number is to another."

If his first arithm was x, he then boldly grabs another arithm—let's call it z—and imagines cubes with sides $(z+1)x$ and zx, respectively. A bit of standard algebra shows $(3zz+3z+1)xx = 1$, and therefore $3zz+3z+1$ should be a square number. Diophantus assumes it to be the square of $(2z-1)$—how does he get away with that?—and then finds $z = 7$. He now repeats his initial argument with 7 arithms and 8 arithms, and finds the arithm to be 1/13. In our language: $x = 7/13$ and $y = 8/13$.

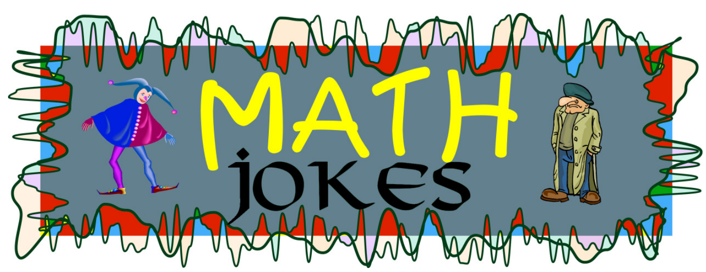
Is this a solution? Yes. Is it the general solution? No. But it points to a technique: had he taken $(z+2)x$ and zx, he would, in the same way, have obtained $6zz + 12z + 8$ and concluded that it should be twice a square number. Setting it equal to twice the square of $(3z-2)$ would have yielded $z = 3$ and the arithm 1/7. In modern language: $x = 3/7$ and $y = 5/7$. There is method in this madness. Can you discover it?

We've discussed enough for today, but this is not the end of Kasparov's intellectual challenges to scholars and his questioning of widely accepted theories. They certainly have taken us on an interesting journey—and left us much to ponder.

If you are interested in learning more about issues relating to chronology, we invite you to visit the discussion forum at the web site

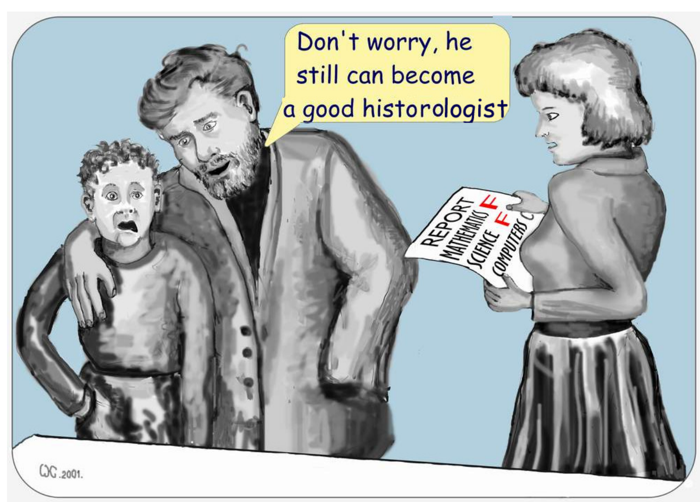
http://www.revisedhistory.org/forum.

Garry Kasparov, the author of the article "Mathematics of the Past" on page 5, will check this site periodically and try to respond to your questions. Submissions will be moderated before publication in the Forum.



Q: What does the math PhD with a job say to the math PhD without a job?

A: "Paper or plastic?"



Math Discovery

Decoding Dates from Ancient Horoscopes

by
Wieslaw Krawcewicz

Mysterious celestial objects visible in the sky have always fascinated and inspired humanity. Even today, in this age of super-rationality and high technology, in spite of its evident groundlessness, astrology seems to preoccupy many people who strongly believe in the supernatural influence of the planetary movements on human lives. Since ancient times, the sky has been believed to be a gate to the Heavens. The changing positions of the planets, the moon, and the sun were seen as expressions of a divine power influencing human existence on Earth. Great importance was attributed to all celestial phenomena, in particular to horoscopes. Regardless of the imaginary significance attributed to horoscopes, we should remember that they are also a record of dates written by means of a cosmic calendar. Today, we can decode ancient horoscopes and, using mathematical computations, discover the dates that were commemorated.

But what exactly is a horoscope? When we look at the sky at night, we get the impression the Earth is surrounded by an enormous sphere filled with stars. Although this celestial sphere seems to be revolving slowly around us (an illusion caused by the daily revolution of the Earth), the stars always appear in the same configurations (called constellations), at the same fixed positions on the celestial sphere. However, there are also other celestial objects, which seem to be "travelling" across the celestial sphere. The moon is one of them, but there are also five planets that can be observed with the naked eye. These planets are Jupiter, Saturn, Mars, Venus, and Mercury. Of course, although invisible at night, the sun is also moving across the sky.

The planets, including the moon and sun, were in old times called travelling stars, but today we simply call them the seven planets of antiquity. It appears to an Earth-based observer that in the course of one year, the sun completes a full revolution around a large circle on the celestial sphere. This circle is called the ecliptic. The planets and the moon are always found in the sky within a narrow belt, 18° wide, centered on the ecliptic, called the zodiac. The area around it is called the zodiacal belt. The zodiacal belt is a celestial highway where the movement of the planets, the sun, and the moon takes place when observed from the Earth. Twelve constellations along the ecliptic comprise the zodiac belt. Their familiar names are Aries, Taurus, Gemini, Cancer, Leo, Virgo, Libra, Scorpio, Sagittarius, Capricorn, Aquarius, and Pisces. Each of the 12 zodiac constellations is located in a sector 30° long, on average (see Figure 1).

The key concept in astrology is a horoscope, which is a chart showing the positions of the planets in the sky with respect to the zodiac constellations. In ancient times, people attributed great importance to these planetary positions and unknowingly encoded in horoscopes the exact dates related to astronomical events.

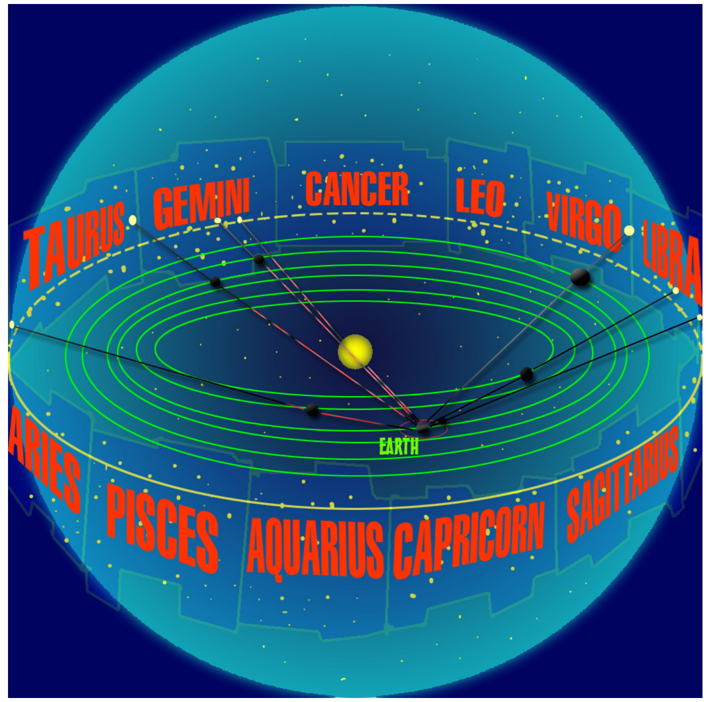


Figure 1
Celestial sphere with the solar system inside. To an Earth-based observer, the planets, the sun, and the moon appear on the zodiacal belt. Their positions are changing continuously.

An astronomical situation shown in a horoscope is quite unique. At any time, there are 12 possible zodiac constellations, where each of the seven "planets" may appear (see Figure 1). The positions of the moon, the sun, Mars, Jupiter, and Saturn are independent of each other. However, due to the inner orbits with respect to the Earth's orbit, the visual angle distance from Mercury to the sun cannot be larger than 28°, and the angle distance from Venus to the sun must be smaller than 48°. This means that for each fixed position of the sun in the zodiac, there are only three possible positions for Mercury and five possible positions for Venus. It is not difficult to compute that there are exactly

$$12 \times 12 \times 12 \times 12 \times 12 \times 3 \times 5 = 3\ 732\ 480$$

different horoscopes. Since an average horoscope remains in the sky for about 24 hours, there are about 365 different horoscopes every year. Therefore, a specific horoscope should reappear only after 10 000 years, on average. However, in reality, a horoscope may reappear more often. The existence of so-called false periods has been observed by researchers.[1] It appears that two or three repetitions of the same horoscope are possible in a period of about 2 600 years, but later such a horoscope disappears for many dozens of thousands of years.

With the use of modern computational methods, it is possible to calculate all of the dates that could correspond to such a horoscope. If other astronomical information is also available from the horoscope (such as the order of the planets or their visibility), it is often possible to eliminate all of the dates except one, which is exactly the date of the horoscope. In this way, mathematics can be a very powerful tool in revealing the mysteries of the ancient world.

1 See [5], Vol.6.

There are many ancient representations of zodiacs containing symbolic representations of horoscopes. In particular, some Egyptian zodiacs, which use specific ancient symbols to illustrate astronomical objects, can be analyzed. It would be difficult to disagree that this is an exciting idea, as it could lead us to the exact dates corresponding to ancient Egyptian history!

Let me include some examples of Egyptian zodiacs. All of these zodiacs are discussed in detail in an upcoming book entirely devoted to the astronomical dating of the ancient Egyptian zodiacs.[2] Figure 2 shows an Egyptian zodiac found on the ceiling in an ancient Egyptian temple in Denderah. It is called the Round Denderah zodiac.

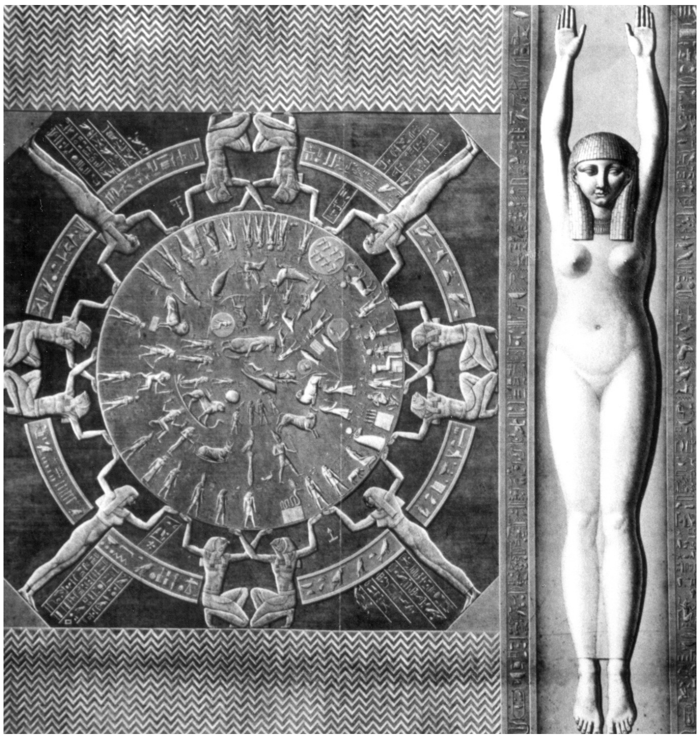


Figure 2

A drawing of the Round Denderah zodiac made during the Napoleonic expedition to Egypt in 1799.[3]

A second zodiac found in the same temple in Denderah is called the Long Denderah zodiac (see Figure 3).

A drawing of another Egyptian zodiac is shown in Figure 4. This zodiac was found in the main hall of a huge temple in the ancient city of Esna, located on a bank of the river Nile. We will call it the Big Esna zodiac.

In the same city of Esna, another zodiac was found by the Napoleonic army in a much smaller temple (see Figure 5). We will call it the Small Esna zodiac, but this name has nothing to do with the size of the zodiac itself.

There are many more Egyptian zodiacs containing horoscopes, but it is not possible to discuss them all in such a short article.[4]

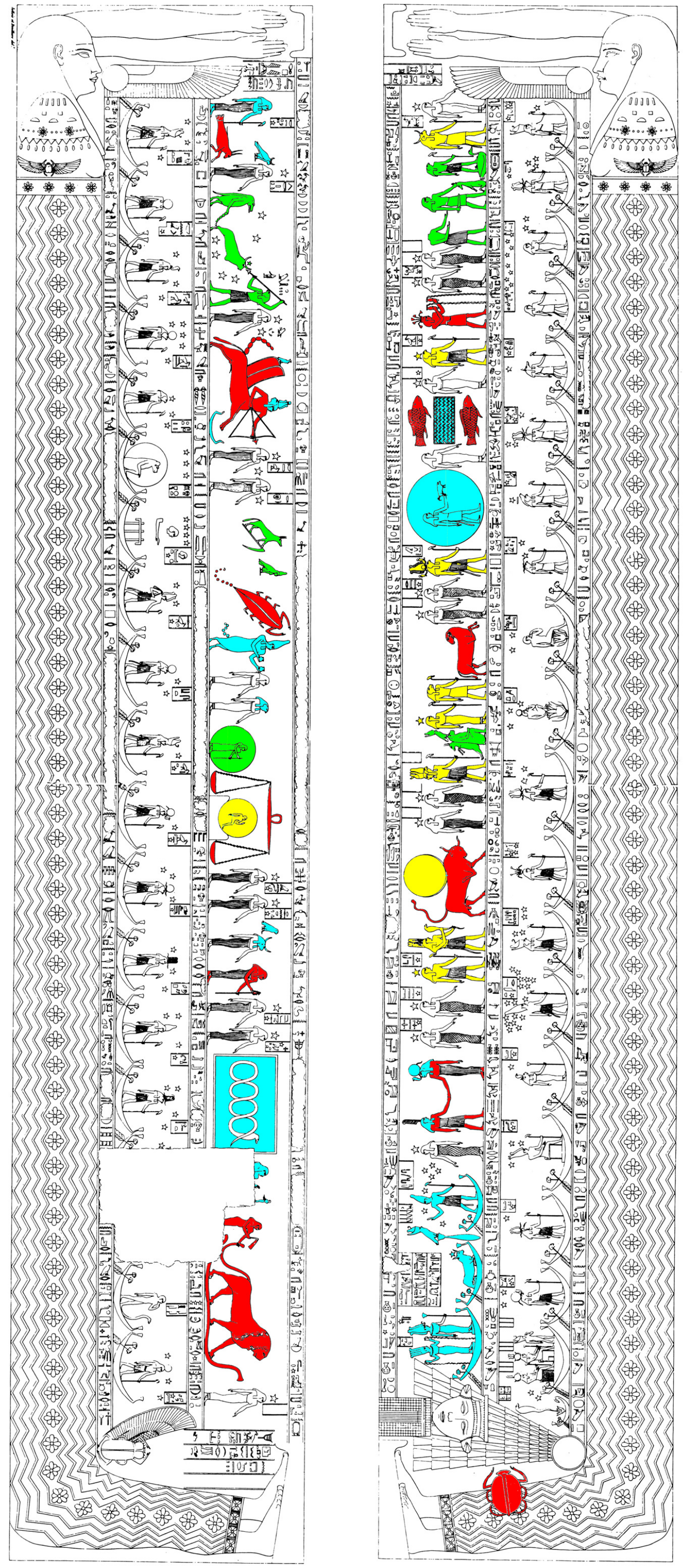


Figure 3

A drawing of the Long Denderah zodiac from the temple in Denderah in Egypt.[5] *Colour annotations were added to indicate constellations (red), planets (yellow), and other astronomical symbols (blue or green).*

[2] See [1].

[3] Picture taken from [2], A. Vol. IV, Plate 21.

[4] For example, there are many zodiacs found inside ancient Egyptian tombs. Read more about it in [1].

[5] Picture taken from [2]. Annotations were made by A.T. Fomenko, T.N. Fomenko, W.Z. Krawcewicz, and G.V. Nosovskij [1].

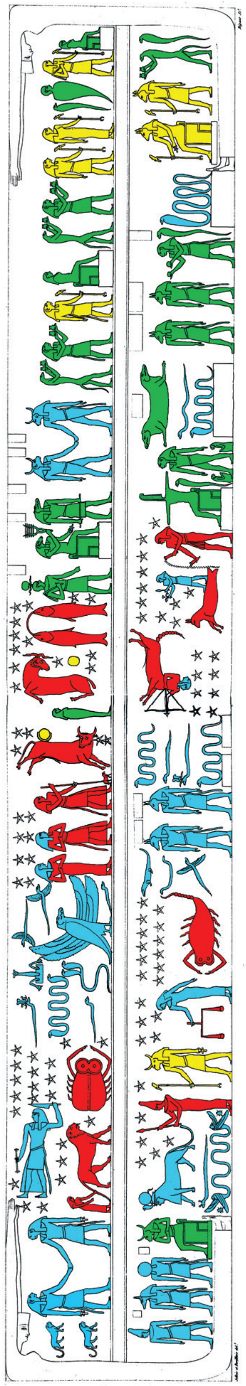


Figure 4

The Big Esna zodiac.[6] *The zodiac constellations are marked in red, the planets in yellow, and the other astronomical symbols in blue and green.*

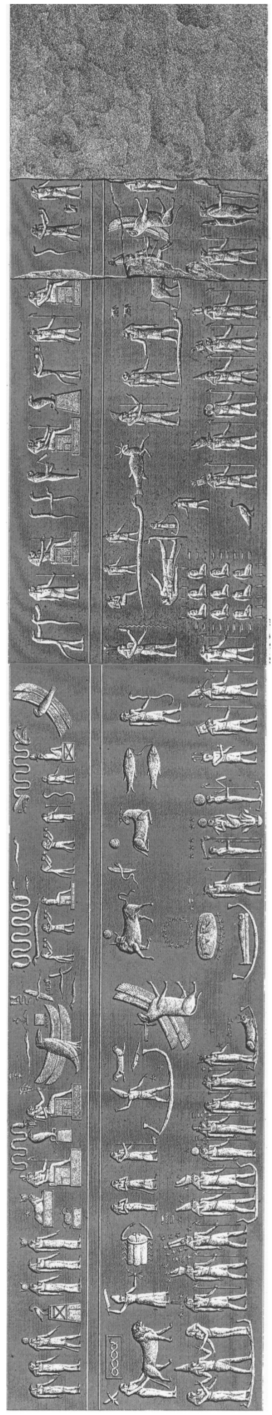


Figure 5

Drawing taken from the Napoleonic Album[7] *of the Small Esna zodiac.*

Egyptian zodiacs should be seen as symbolic representations of astronomical objects inside the zodiacal belt. The actual decoding of the astronomical symbolism of such a zodiac is rather complicated. In Figure 6, we show a drawing of the Round Denderah zodiac taken from the book [1], where it is carefully analyzed and decoded.

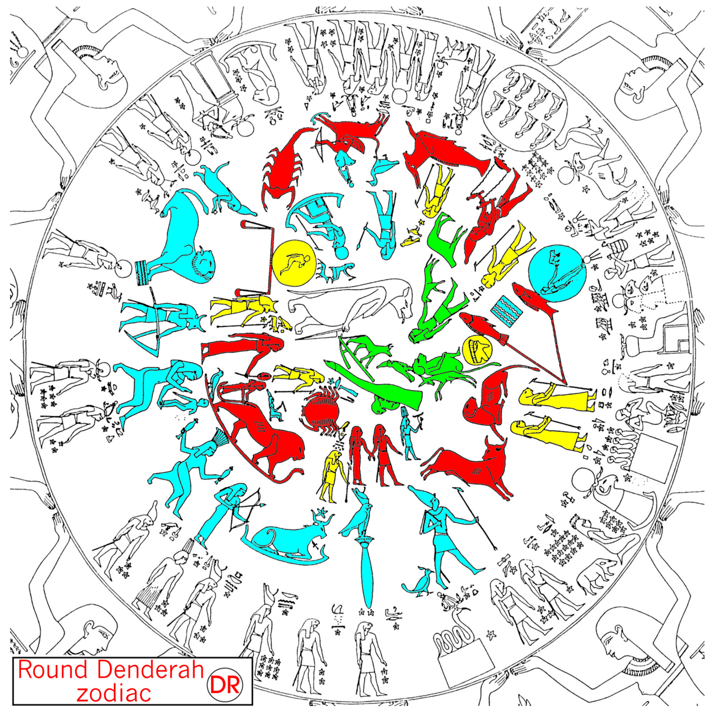


Figure 6

Decoded astronomical meaning of the Round Denderah zodiac. The zodiac constellations are marked in red, the planets in yellow, and the other astronomical symbols in blue and green.

In this representation, colours are used to distinguish figures of different astronomical meaning. The red figures are the zodiac constellations, which can be easily recognized because their appearance has remained largely unchanged to present times. The yellow figures are the planets. Some are marked by hieroglyphic inscriptions, but it is generally not an easy task to determine exactly which planets are represented by these symbols.

The blue and green figures represent other astronomical symbols. The blue colour indicates the astronomical meaning of the figure was successfully decoded, and the green colour indicates the meaning of the figure was not completely understood.

The final decoding was achieved through a complicated elimination process,[8] in which all possible variants were considered. For each of the dates obtained, all of the available astronomical data was carefully verified, and only solutions satisfying all of the required conditions were considered.

It was found that the figures shown on this zodiac indicate that: the moon was in Libra; Saturn was in either Virgo or Leo; Mars was in Capricorn; Jupiter was in either Cancer or Leo; Venus was in Aries; and Mercury and the sun were in Pisces.

Dating of this zodiac was done using the astronomical software HOROS, which was developed by Russian mathematician

[6] Picture taken from [2], A. Vol. I, Plate 79. Colour annotations were made by A.T. Fomenko, T.N. Fomenko, W.Z. Krawcewicz, and G.V. Nosovskij [1].

[7] Picture taken from [2], A. Vol. I, Plate 87.

[8] See [1].

G.V. Nosovskij, based on an algorithm used by the French astronomers J.L. Simon, P. Bretagon, J. Chapront, M. Chapront, G. Francou, and J. Laskar, in an astronomical program called PLANETAP.[9]

This software, together with sample input files and brief instructions, is available at the π *in the Sky* web site:

http://www.pims.math.ca/pi/.

The results presented in [1] are most intriguing. The dates obtained are as follows:

- **Round Denderah zodiac:** morning of **March 20, 1185 A.D.**
- **Long zodiac: April 22–26, 1168 A.D.**
- **Big Esna zodiac: March 31–April 3, 1394 A.D.**
- **Small Esna zodiac: May 6–8, 1404 A.D.**

Of course, these dates completely contradict the chronology of ancient Egypt and have created a controversy regarding the age of the ancient Egyptian monuments. But still, the results stand for themselves. Clearly more research is needed before final conclusions can be drawn.

References:

[1] A.T. Fomenko, T.N. Fomenko, W.Z. Krawcewicz, G.V. Nosovskij. Mysteries of Egyptian Zodiacs and Other Riddles of Ancient History. To appear.

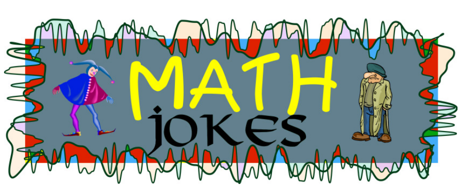
[2] Description de l'Egypte. Publiée sous les ordres de Napoléon Bonaparte, Bibliothèque de l'Image, Inter-Livres, 1995.

[3] Ancient Egypt. General Editor David P. Silverman, New York Oxford University Press, 1997.

[4] Simon J.L., Bretagnon P., Chapront J., Chapront-Touze M., Francou G., Laskar J., Astron. Astrophys, 282, 663 (1994).

[5] N.A. Morozov. Christ. The History of Human Culture from the Standpoint of the Natural Sciences. (In Russian), Moscow and Leningrad. 1926–1932, vols. 1–7. Second edition, Kraft & Lean, Moscow, 1997–1998, vols. 1–7.

If you have any comment, remark or question related to this article, or you would like to share your opinion, send your email directly to **Wieslaw Krawcewicz** at wieslawk@shaw.ca.



A statistics professor plans to travel to a conference by airplane. When he passes the security check, a bomb is discovered in his carry-on baggage. Of course, he is hauled off immediately for interrogation.

"I don't understand it!" the interrogating officer exclaims. "You're an accomplished professional, a caring family man, a pillar of your parish—and now you want to destroy all that by blowing up an airplane!"

"Sorry," the professor interrupts him. "I never intended to blow up the plane."

"So, for what reason did you try to bring a bomb on board?!"

"Let me explain. Statistics show that the probability of a bomb being on an airplane is 1/1000. That's quite high if you think about it—so high that I wouldn't have any peace of mind on a flight."

"And what does this have to do with *you* bringing a bomb on board?"

"You see, since the probability of *one* bomb being on my plane is 1/1000, the chance that there are *two* bombs is 1/1 000 000. So, if I already bring one, I am much safer..."

A physics professor conducting experiments has worked out a set of equations that seem to explain his data. Nevertheless, he is unsure if his equations are really correct and therefore asks a colleague from the math department to check them.

A week later, the math professor calls him: "I'm sorry, but your equations are complete nonsense."

The physics professor is, of course, disappointed. Strangely, however, his incorrect equations turn out to be surprisingly accurate in predicting the results of further experiments. So, he asks the mathematician if he was sure about the equations being completely wrong.

"Well," the mathematician replies, "they are not actually complete nonsense. But the only case in which they are true is the trivial one in which the time variable is supposed to be a non-negative real number..."

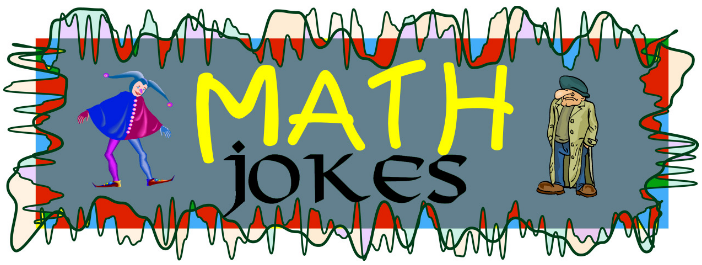
A physicist, a mathematician, and a computer scientist discuss which is better: a husband or a boyfriend.

The physicist: "A boyfriend. You still have freedom to experiment."

The mathematician: "A husband. You have security."

The computer scientist: "Both. When I'm not with my husband, he thinks I'm with my boyfriend. When I'm not with my boyfriend, it's vice versa. And I can be with my computer without anyone disturbing me..."

[9] See [4].



"Wasn't yesterday your first wedding anniversary? What was it like being married to a mathematician for a whole year?'

"She just filed for divorce..."

"I don't believe it! Did you forget about your anniversary?"

"No. Actually, on my way home from work, I stopped at a flower store and bought a bouquet of red roses for my wife. When I got home, I gave her the roses and said 'I love you'."

"So, what happened?!"

"Well, she took the roses, slapped them around my face, kicked me in the groin, and threw me out of our apartment..."

"I can't believe she did that!!"

"It's all my fault... I should have said 'I love you *and only you*'."

"Statistics show that most people are deformed!"

"How is that?"

"According to statistics, an average person has *one* breast and *one* testicle..."

A mathematician, a physicist, and an engineer are asked to test the following hypothesis: All odd numbers greater than one are prime.

The mathematician: "Three is a prime, five is a prime, seven is a prime, but nine is not a prime. Therefore, the hypothesis is false."

The physicist: "Three is a prime, five is a prime, seven is a prime, nine is not a prime, eleven is a prime, and thirteen is a prime. Hence, five out of six experiments support the hypothesis. It must be true."

The engineer: "Three is a prime, five is a prime, seven is a prime, nine is a prime..."

Psychologists subject an engineer, a physicist, and a mathematician—a topologist, by the way—to an experiment: Each of them is locked in a room for a day—hungry, with a can of food, but without an opener; all they have is pencil and paper.

At the end of the day, the psychologists open the engineer's room first. Pencil and paper are unused, but the walls of the room are covered with dents. The engineer is sitting on the floor and eating from the open can: He threw it against the walls until it cracked open.

The physicist is next. The paper is covered with formulas, there is one dent in the wall, and the physicist is eating, too: he calculated how exactly to throw the can against the wall, so that it would crack open.

When the psychologists open the mathematician's room, the paper is also full of formulas, the can is still closed, and the mathematician has disappeared. But there are strange noises coming from inside the can...

Someone gets an opener and opens the can. The mathematician crawls out. "Darn! I got a sign wrong..."

Isn't math poetic?

$$\int_1^{\sqrt[3]{3}} v^2\, dv\, \cos\left(\frac{3\pi}{9}\right) = \log \sqrt[3]{e}.$$

In words:

The integral v squared dv
From 1 to the cube root of 3
Times the cosine
Of 3π over 9
Is the log of the cube root of e.

When the logician's little son refused again to eat his vegetables for dinner, the father threatened him: "If you don't eat your veggies, you won't get any ice cream!"

The son, frightened at the prospect of not having his favourite dessert, quickly finished his vegetables.

What happened next?

After dinner, impressed that his son had eaten all of his vegetables, the father sent his son to bed without any ice cream...

Q: Why does a chicken cross a Möbius strip?
A: To get to the same side.

Q: How do you call a one-sided nudie bar?
A: A Möbius strip club!

A western military general visits Algeria. As part of his program, he delivers a speech to the Algerian people: "You know, I regret that I have to give this speech in English. I would very much prefer to talk to you in your own language. But unfortunately, I was never good at algebra..."

Q: What do you call the largest accumulation point of poles?
A: Warsaw!

A math professor is talking to her little brother who just started his first year of graduate school in mathematics.

"What's your favourite thing about mathematics?" the brother wants to know.

"Knot theory."

"Yeah, me neither!"

Solar Eclipses: Geometry, Frequency, Cycles

by

Hermann Koenig[†]

Total solar eclipses are spectacular shows in the sky, in particular, if they occur on a bright day around noon. In a narrow band on Earth, the moon completely obscures the sun and the solar corona becomes visible. Sun and moon both appear to the observer on Earth to subtend an angle of roughly $\theta = \frac{1}{2}^\circ$, even though the radius of the sun is 400 times larger than that of the moon. By pure chance, the sun is also about 400 times further away from the Earth than the moon. The values are

$$\theta_S \approx 2 \sin \frac{\theta_S}{2} = \frac{2R}{D}, \quad \theta_M \approx 2 \sin \frac{\theta_M}{2} = \frac{2r}{d} \quad \text{(in radians)}.$$

Notation	Value	Meaning
R	696 000 km	Radius of the sun
r	1 738 km	Radius of the moon
D	149 600 000 km	Distance from the sun to the Earth (mean value)
d	384 400 km	Distance from the moon to the center of the Earth (mean value)
$\theta_S(\theta_M)$		Apparent angle of the sun (moon), as seen from the surface of the Earth

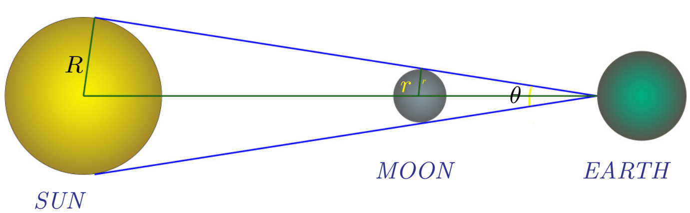


Figure 1

Angles of the sun and the moon (not to scale).

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Since $\frac{R}{r} \approx 400 \approx \frac{D}{d}$, we see that $\theta_S \approx \theta_M$. The sun is at one focus of the counterclockwise elliptical orbit of the Earth around the sun. Thus, the distance D between the sun and the Earth actually varies between 147 100 000 km (perihelion, which occurs each year around January 3) and 152 100 000 km (aphelion, which occurs around July 4). The moon's distance from the center of the Earth varies even more percentage-wise, between 357 300 km and 406 500 km (in the new moon position). Thus, the actual values of the angles θ_S and θ_M range as follows:

$$0.524^\circ \leq \theta_S \leq 0.542^\circ, \qquad 0.497^\circ \leq \theta_M \leq 0.567^\circ,$$

with mean values over time of $\theta_S = 0.533^\circ$ and $\theta_M = 0.527^\circ$. If the moon does not cover the sun completely, an annular or partial eclipse may result.

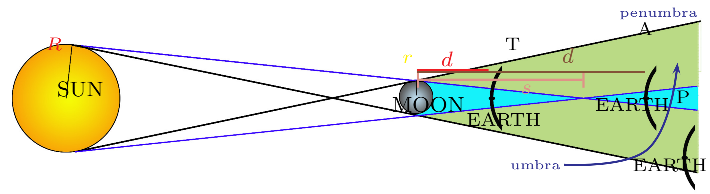


Figure 2

Three positions of Earth: Total (T), Annular (A), and Partial (P) eclipses.

Figure 2 illustrates the positions of the sun, moon, and Earth during solar eclipses, though not to scale. They are in line, with the moon in the new moon position. The very narrow ($\frac{1}{2}^\circ$ wide) shadow cone of the moon, the umbra, has its vertex at a distance s from the moon in the general direction of the Earth. We calculate, looking at Figure 2, that

$$\frac{r}{s} = \frac{R}{s + D - d}\,, \quad s = \frac{(D-d)r}{R-r} \approx \frac{Dr}{R} \approx \frac{D}{400.5};$$

hence

$$367\ 300 \text{ km} \leq s \leq 379\ 800 \text{ km}.$$

Thus, depending on the moon's distance, the Earth's surface can be on either side of the umbral vertex ("shadow boundary"). In position T, for $d < s$, a total solar eclipse occurs. If d is larger, $d > s$, the Earth is in position A, in the inverted umbral cone and the observer on Earth sees an annular eclipse; the moon obscures the central part of the sun but not the fringes. If the Earth is in the penumbra, a partial eclipse occurs.

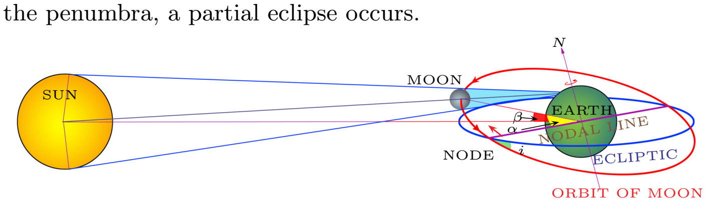


Figure 3

Sun, Earth, and moon: Angles α and β have to be small in the case of an eclipse (section of the Earth perpendicular to the ecliptic).

Shouldn't there then be a solar eclipse at every new moon, one every lunar month, after $T_{\text{lun}} = 29.531$ days (on average)? This would be the case if the moon were to orbit the Earth in the ecliptic (the plane of Earth's elliptical movement around the sun). However, the orbital planes are inclined to each another by $i = 5.14^\circ$. They intersect along the nodal line (see Figure 3). Solar eclipses occur when the moon crosses this line in the decreasing or increasing node (north to south or south to north), or is close to the node when the nodal line points toward the sun: eclipses require the moon to be in or near the ecliptic (thus the name!). If the moon is not close to either node, it will be too far north or south of the ecliptic; its narrow shadow cone will miss the Earth. Let β be the angle between the moon–Earth and sun–Earth lines in the new-moon position. For $\beta > 0.95°$, the center of the moon is more than

$$0.95° \times \frac{\pi}{180} \times 384\,000 \text{ km} \approx 6370 \text{ km} = R_0$$

away from the ecliptic. Because of the sun's large distance from the moon and the Earth, the central line of the moon's shadow is practically parallel to the ecliptic and will miss the Earth, which has the radius R_0. Therefore, no central (i.e., either total or annular) eclipse will occur. The angle β is directly related to the angle α between the moon–Earth line and the nodal line in the moon's orbital plane. Figure 4 and a little bit of spherical trigonometry give the formula

$$\sin\alpha = \frac{\sin\beta}{\sin i}.$$

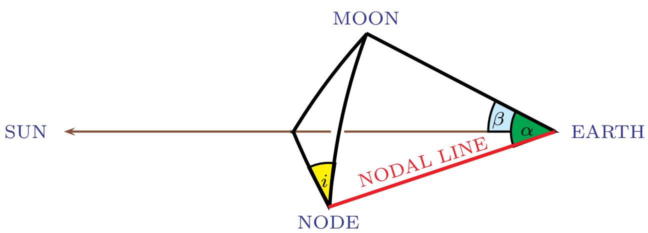


Figure 4

The spherical triangle: node, moon, sun (positions and angles as in Figure 3).

Therefore, if $|\alpha| \le 10.5°$, then $|\beta|$ is $\le 0.95°$, and a central eclipse will occur (slightly different values are possible since D and d vary). Similarly, for $|\beta| \le 1.4°$ and $|\alpha| \le 16°$, at least partial eclipses will occur. In this case, in the above argument the Earth's radius R_0 has to be replaced by R_0 plus the fairly large radius (more than 3000 km) of the penumbra, giving the larger bound for beta.

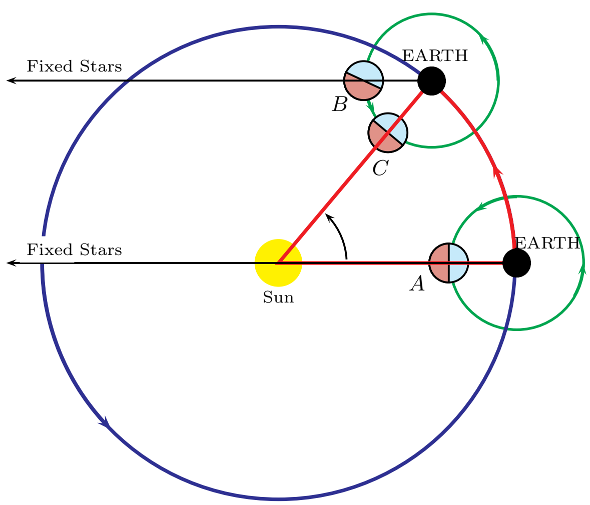


Figure 5

Lunar and sidereal months: (A) new moon, (B) one sidereal month later, and (C) one lunar month later (distances and angles are not to scale).

It takes the moon only $T_{\text{sid}} = 27.322$ days on average to orbit the Earth once with respect to the fixed stars. This is the sidereal month. During this time, the Earth progresses in its orbit around the sun. Hence, the moon needs more time to move from one new moon position to the next; this is the previously mentioned lunar month T_{lun} (see Figure 5). The sidereal and lunar months are related as follows:

$$\frac{1}{T_{\text{sid}}} = \frac{1}{T_{\text{lun}}} + \frac{1}{J},$$

where $J = 365.242$ days, the length of the (tropical) year.

The period of time the moon needs to move from one descending node to the next, the so-called draconic month $T_{\text{dr}} = 27.212$ days, is even shorter then the sidereal month since the nodal line rotates clockwise once every 18.62 years around the Earth, in a gyroscopic effect caused by the sun and Earth. So how often do eclipses occur? In one draconic month's orbit, four angular sectors of 10.5°, two on each side of the two nodes, are favourable for central eclipses if the moon is positioned there. Hence, on average, a total or annular solar eclipse will happen every

$$\frac{360°}{4 \times 10.5°} \times 27.21 \text{ days} = 233 \text{ days}$$

somewhere on Earth, which means 156 per century. As for (at least) partial eclipses, the frequency is one every

$$\frac{360°}{4 \times 16°} \times 27.21 \text{ days} = 153 \text{ days},$$

or 238 per century. These numbers agree with long-time statistics of solar eclipses. Since $\theta_M < \theta_S$ holds on average, annular eclipses slightly outnumber total eclipses: of those 156 central eclipses, about 65 are total, 78 are annular, and 13 are mixed. Total eclipses are more likely to occur in summer (June to August) since the sun is close to its aphelion and appears to us at the smallest possible angle. Annular eclipses dominate during the northern-hemisphere winter. Since the Earth's axis is tilted toward the sun during the summer, total eclipses are slightly more frequent in the northern hemisphere of the Earth than in the southern; the opposite holds true for annular eclipses.

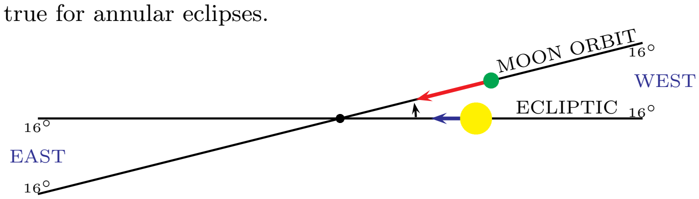


Figure 6

The "danger zone" of an eclipse near a node: the moon passing the sun near the descending node (view fixing the node).

From the Earth, let us look toward a node of the moon's orbit and the ecliptic, when the sun and the moon are in the ±16° sectors around the node. The (say) descending node points every

$$J_{\text{ecl}} = \frac{365.242}{1 + 1/18.62} \text{ days} = 346.62 \text{ days}$$

toward the sun: this is the ecliptic year. Here 365.242 days is the length of the (tropical) year. Therefore, the sun needs

$$\frac{2 \times 16°}{360°} \times 346.62 \text{ days} = 30.8 \text{ days}$$

to pass the "danger zone" of an eclipse. Since this is more than a lunar month, the moon will overtake the sun at least once, maybe twice, during this time. This results in one or two solar eclipses every half ecliptic year. We conclude, therefore, that every year there are at least two and at most five solar eclipses (total, annular or partial) per year *somewhere on Earth*. The fifth eclipse may occur in the "leftover" 18.62 days $(365.24 - 346.62)$, although this is a rather rare event, happening the next time in the year 2206. Typically, one total and one annular eclipse or two or four partial eclipses occur in a given year; this being the case, for example, in 2002, 2004, and 2000, respectively.

At a *specific location*, say Edmonton, Calgary, or Vancouver, a total solar eclipse is quite rare, with one happening about every 390 years *on average*. This figure, however, is subject to large variations. For example, a coastal strip in Angola is the scene of two total solar eclipses in 2001 and 2002, whereas London did not experience any total solar eclipses between 878 A.D. and 1715 (when Halley produced the first eclipse map).

Total solar eclipses are favoured if the moon's distance d is small (close to its perigee) and the sun's distance D is large (close to its aphelion). A small sun is covered by a large moon. The time between two successive perigees of the moon (closest distance points) is the anomalistic month $T_{\text{an}} = 27.555$ days. It is larger than the draconic month since the perigee moves slowly in a counterclockwise direction under the influence of the sun, completing one rotation in 8.85 years. Interestingly, there are good rational approximations of the ratios of these different types of months: 223 lunar months are almost the same as 242 draconic months, 19 ecliptic years, or 239 anomalistic months. This is the Saros period S:

$$\begin{aligned} S = 223\ T_{\text{lun}} &= 6585.32 \text{ days}, \\ 242\ T_{\text{dr}} &= 6585.36 \text{ days}, \\ 19\ J_{\text{ecl}} &= 6585.78 \text{ days}, \\ 239\ T_{\text{an}} &= 6585.54 \text{ days}. \end{aligned}$$

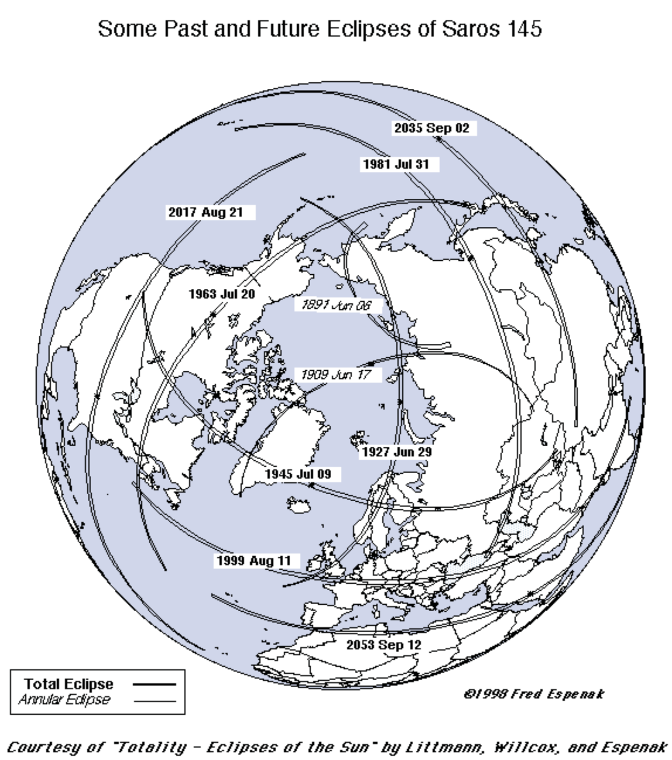
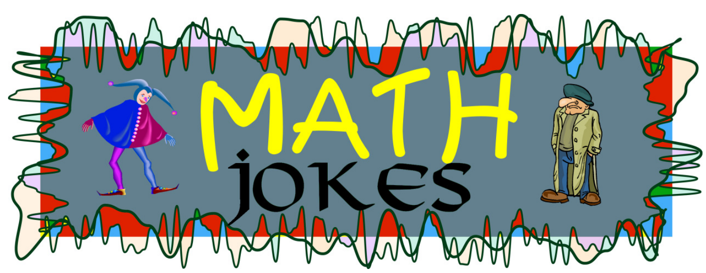


Figure 7

Paths of nine successive total solar eclipses in the Saros series 145.

In years, S is 18 years plus $10\frac{1}{3}$ or $11\frac{1}{3}$ days, depending on the number of leap years during this time. Solar eclipses thus tend to repeat after the period S: the moon is again in the new moon position and in the same type of node (decreasing/increasing) pointing toward the sun, the Earth–sun distance is about the same after almost 18 years and the Earth–moon distance is very similar after 239 anomalistic months. This means that the type of eclipse typically is the same (annular, total, or partial): the latitude of the tracks of central eclipses on Earth is only slightly shifted north/south, but the longitude of the next eclipse in a Saros cycle is $0.32 \times 360^\circ = 115^\circ$ further to the west. After three such periods, 54 years and one month, a very similar eclipse will reappear in almost the same longitude, latitude being somewhat shifted north or south. Since the above periods do not coincide perfectly, any such Saros series of eclipses eventually ends after about 72 eclipses, which move slowly in about 1 300 years from the south to the north pole or vice-versa. Figure 7 shows the paths of nine successive total solar eclipses in the Saros series 145; the calculations were done by Fred Espenak (NASA/GSFC).

So when is the next total solar eclipse in Western Canada? You will have to wait until the afternoon of August 22, 2044, when an eclipse will be experienced in Edmonton and Calgary, with the sun being at an altitude of 10° above the horizon. This eclipse will have predecessors in its Saros series in 2008 in Siberia/China (around the time of the Beijing Olympic Games) and 2026 in the North Atlantic and Spain. The last total eclipse in Edmonton was in 1433—a gap of more than 600 years, although in 1869 there was one visible just south of the city area. Banff, Calgary, Ohio, and Virginia will experience another total eclipse in September 2099. This one will have its Saros predecessors in 2045 in the U.S., tracking from Oregon to Florida, and in 2009 in Shanghai, China and in the western Pacific. The 2009 occurrence will be the most massive total eclipse of the 21$^{\text{st}}$ century, with a totality phase lasting up to $6\frac{2}{3}$ minutes. The 1999 eclipse in central Europe will be followed in its Saros series 145 by a total eclipse in August 2017 in the U.S., tracking from Oregon to South Carolina. This is the next total solar eclipse in the U.S.; its totality phase will last up to $2\frac{1}{2}$ minutes.



A mathematician has spent years trying to prove the Riemann hypothesis, without success. Finally, he decides to sell his soul to the devil in exchange for a proof. The devil promises to deliver a proof within four weeks.

Four weeks pass, but nothing happens. Half a year later, the devil shows up again—in a rather gloomy mood.

"I'm sorry," he says. "I couldn't prove the Riemann hypothesis either. " But"—and his face lightens up—"I think I found a really interesting lemma. . . "

"The number you have dialed is imaginary. Please, rotate your phone by 90 degrees and try again. . . "

Noether

Volker Runde[†]

The highest honour that can be bestowed upon a mathematician is not the Fields Medal—it is becoming an adjective: Euclid was immortalized in Euclidean geometry; Descartes' memory is preserved in Cartesian coordinates; and Newton lives on in Newtonian mechanics. Emmy Noether's linguistic monument is the Noetherian rings.[1] To my knowledge, Emmy Noether is still the only female mathematician ever to have received this honour, and she definitely was the first.

Emmy Noether as young woman.

Amalie "Emmy" Noether was born on March 23, 1882, in the city of Erlangen, in the German province of Bavaria. She was the first child of Max Noether and his wife Ida, née Kaufmann. The math gene ran in her family—her father was a math professor at the University of Erlangen, and her younger brother Fritz would later become a mathematician, too.

Getting a real education was not easy for a woman in those days. In order to be formally enrolled at a German university, you needed (and still need) the *Abitur*, a particular type of high-school diploma, and in those days, there were no schools that allowed girls to graduate with the *Abitur*. Emmy attended a *Höhere Töchter–Schule* in Erlangen, a school that provided the daughters of the bourgeoisie with an education that was deemed suitable for girls (i.e., with an emphasis on languages and the fine arts). Science and mathematics were not taught in any depth. After graduation in 1897, Emmy continued to study French and English privately and, three years later, she passed the Bavarian state exam that allowed her to teach French and English at girls' schools.

Instead of working as a language teacher, Emmy spent 1900 to 1903 auditing lectures at the University of Erlangen in subjects such as history, philology, and—of course!—mathematics. During the same period, she also started preparing for the *Abitur* exam. In July 1903, Emmy obtained her *Abitur*. Fortunately, at about the same time, women who had the *Abitur* were allowed to enroll officially at Bavarian universities. After a year at Göttingen, she enrolled at Erlangen in 1904, where she obtained her doctorate in 1907. Although, at that time, it was already quite an accomplishment for a woman to obtain a doctorate at all, her thesis did little to indicate her future stature as a mathematician. Later in life, Emmy herself referred to her thesis in terms such as "Rechnerei,"[2] "Formelgestrüpp,"[3] and even "Mist."[4]

Emmy Noether as a budding mathematician.

Having received her doctorate, Emmy continued working as a mathematical researcher in Erlangen, without a position and, of course, without pay. Nevertheless, in the following years she built a reputation as mathematician, and in 1909, she was the first woman invited to speak at the annual congress of the German Mathematical Association. In 1915, the mathematicians Felix Klein and David Hilbert invited her to join them at Göttingen, in the province of Prussia. To say that Göttingen was the center of the mathematical world in those days would be an understatement. From the 19th century to the early 1930s, Göttingen simply *was* the mathematical world. Needless to say, Emmy went.

Informal photo of Emmy Noether.

The invitation to Göttingen was not a job offer. There were serious legal obstacles to becoming a professor. In Germany, a doctorate is not sufficient qualification to become a professor—it is necessary to obtain the *Habilitation*, which is a kind of second, more demanding doctorate. Without doubt, Emmy Noether was a strong enough mathematician to obtain it, but Prussian law at that time admitted only males as candidates for the *Habilitation*. When Emmy, supported by David Hilbert, nevertheless filed for her *Habilitation*, the Prussian Ministry of Culture intervened and forbade it. Despite this setback, she started teaching at Göttingen in 1915, albeit without pay and not under her own name—officially, Hilbert was the instructor, and she only assisted him.

In 1919, many things changed in Europe. In Germany, the monarchy was overthrown. For the first time, women had the right to vote, and Emmy Noether could finally file for the *Habilitation* without legal problems. This didn't mean that suddenly everything became easy for her (or for other women at German universities). Her *Habilitation* had to be approved by the university's senate. The very thought of a woman receiving the *Habilitation* stirred a heated debate in the senate. "If we grant her the *Habilitation*," her opponents argued, "then she might one day become a professor, and if she becomes a professor, she might be elected into the university's senate, and the idea of a woman sitting in the senate is so abhorrent that it must not be allowed to happen."

[†] **Volker Runde** is a professor in the Department of Mathematical Sciences at the University of Alberta. His web site is `http://www.math.ualberta.ca/~runde/runde.html` and his *e-mail* is `vrunde@ualberta.ca`.

[1] I won't attempt to explain what they are, but you will encounter them when you take your first course in abstract algebra.

[2] mere computations

[3] shrub of formulas

[4] manure

I'm not kidding—such arguments were seriously brought forward against Emmy Noether's *Habilitation*. According to legend, it was David Hilbert who ended the debate in the senate with the memorable phrase: "Meine Herren! Der Senat ist keine Badeanstalt!"[5] As the first woman at a German university, Emmy Noether was granted the *Habilitation* in mathematics and could finally lecture under her own name (but still without pay).

In 1922, the University of Göttingen granted Emmy Noether the title of Extraordinary Professor. The extraordinary thing about this professorship was that it gave Emmy the right to call herself a professor, but did not come with any salary. Until 1921, Max Noether had financially supported his daughter. When he died, her financial situation became tight. In order to prevent Emmy from becoming destitute, the university finally gave her a paid teaching assignment in 1923. For the first time in her life, at age 41, Emmy Noether, one of the leading mathematical researchers in Germany (if not in the world), had an income of her own.

Emmy Noether in Göttingen with friends and colleagues in the 1920s.

For the next 10 years, Emmy enjoyed scholarly and professional success. At that time, a revolution took place in mathematics. Emphasis shifted from computations and explicit constructions to more abstract and conceptual approaches. David Hilbert was at the forefront of this revolution—and so was Emmy Noether. In the 1920s and 1930s, the field of algebra changed almost beyond recognition. In 1900, algebra was about solving algebraic equations. Fifty years later, it had become the study of algebraic structures, such as groups, rings, and fields; it had become abstract algebra. One of the driving forces behind this shift was Emmy Noether. A former student of hers, Bartel van der Waerden, a Dutchman, later wrote a textbook on abstract algebra that is, to a large extent, based on Emmy Noether's lectures at Göttingen. It has been a standard text for decades.

By all accounts, Emmy Noether was very popular with students. Although she was demanding and not a very good lecturer, she had a kind personality that compensated for all that. She gathered a group of devoted followers around her—nicknamed her satellites—with whom she went swimming in the municipal pool and who she invited to her small apartment for large bowls of pudding. Of course, besides swimming and eating pudding, Emmy and her satellites had long discussions about mathematics. It is interesting to note how her students dealt with her being a woman—they chose to ignore it. For example, she was referred to as *der Noether*. In German, *der* is the definite article for the masculine gender.


Emmy Noether in her later years.

Every four years, mathematicians hold the International Congress of Mathematicians (ICM). To give an invited address at such a congress is a feat few mathematicians ever accomplish. In 1932, Emmy Noether delivered an invited presentation at the ICM in Zurich. As you may have guessed, she was the first woman to do so—and for a long time after 1932, she would remain the only one. By the way, she still didn't have permanent employment at that time, even though some of her former students had already become professors.

On January 30, 1933, Paul von Hindenburg, president of Germany, appointed Adolf Hitler as chancellor. A month later, the *Reichstag* went up in flames; the government blamed the communists. Three weeks later, the first concentration camp began operating, and on April 1, the systematic persecution of the Jews started with a boycott of Jewish businesses and other facilities.


Emmy Noether in Göttingen in the spring of 1931.

Emmy Noether's political views were left-wing. She espoused pacifism and had been a member of the Social Democratic Party for some time. More significantly, she was Jewish. On April 25, 1933, Emmy Noether was sent on leave, which, in fact, meant that she had been fired.

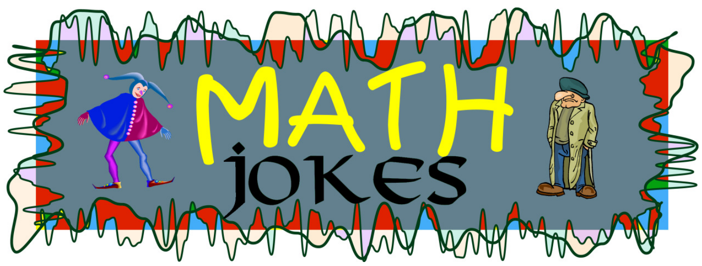
She chose not to take any risks and accepted a visiting position in the U.S., at Bryn Mawr women's college in Pennsylvania. The transition from working mainly as a researcher to undergraduate teaching must have been difficult for her, but she soon succeeded again in surrounding herself with eager young minds. In 1934, Emmy returned to Germany, only to cancel the lease on her

[5] Gentlemen! The senate is not a public swimming pool!

apartment and to arrange to have her belongings shipped back to America. A year later, she underwent brain surgery to have a tumour removed. Even close friends hadn't known of her illness. On April 14, 1935, she died from complications following the operation, never having held a permanent professional position.

As I noted at the beginning of this article, Emmy's younger brother Fritz was also a mathematician. Being male, he enjoyed a smoother progression in his career. Eventually, Fritz became a full professor in Breslau.[6] After the Nazi's rise to power, he decided to leave Germany for the Soviet Union. There, he was one of the millions who disappeared in Stalin's reign of terror.

By the way, if you want to see a Noetherian ring and don't want to wait until you take your first course in abstract algebra, here is an opportunity. There are "Noetherian Rings" at several North American universities (e.g., Berkeley,[7] the University of Florida,[8] and the University of Wisconsin at Madison[9]), each a local organization of women in mathematics—faculty and students, graduate and undergraduate alike.



A math student and a computer science student are jogging together in a park when they hear a voice: "Please, help me!"

They stop and look. The voice belongs to a frog sitting in the grass.

"Please, help me!" the frog repeats. "I'm not really a frog; I'm an enchanted, handsome prince. Kiss me, and the spell will be broken: I will be yours forever..."

The computer science student picks up the frog. She examines it carefully from all sides—not even making an attempt to kiss it.

"You don't have to marry me," the frog continues frantically, "if you're afraid of the commitment. I'll do whatever you wish me to do if you'll just kiss me..."

The frog's voice is silenced when the computer science student puts the animal into her pocket.

"But why don't you kiss him?!" the math student asks.

"You know," she replies, "I simply don't have time for a boyfriend—but a frog that talks makes a really cool pet..."

Q: What is the first derivative of a cow?

A: Prime Rib!

"TWO, THREE, FIVE, SEVEN ... ALL THE PRIME NUMBERS THERE ARE."

A pure and an applied mathematician are asked to calculate the value of two times two.

The applied mathematician's solution: We have

$$2 \cdot 2 = 2 \cdot \frac{1}{1 - \frac{1}{2}}.$$

The second factor on the right-hand side has a geometric series expansion

$$\frac{1}{1 - \frac{1}{2}} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

Cutting off the series after the second term yields the approximate solution

$$2 \cdot 2 = 2 \cdot \left(1 + \frac{1}{2}\right) = 3.$$

The pure mathematician's solution: We have

$$2 \cdot 2 = (-2) \cdot \frac{1}{1 - \frac{3}{2}}.$$

The second factor on the right-hand side has a geometric series expansion

$$\frac{1}{1 - \frac{3}{2}} = 1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \cdots,$$

which diverges. Hence, the solution to $2 \cdot 2$ does not exist.

At a conference, a mathematician proves a theorem.
Someone in the audience interrupts him: "That proof must be wrong—I have a counterexample to your theorem."
The speaker replies: "I don't care—I have another proof for it."

At a press conference held at the White House, president George W. Bush accused mathematicians and computer scientists in the U.S. of misusing classroom authority to promote a Democratic agenda. "Every math or computer science department offers an introduction to AlGore-ithms," the president complained. "But not a single one teaches GeorgeBush-ithms..."

[6] now Wrocław in Poland
[7] http://www.math.berkeley.edu/~nring/
[8] http://www.math.ufl.edu/~nring/
[9] http://www.math.wisc.edu/~hollings/noethring/

"Rare" Events May Not Be So Rare After All

Carl Schwarz†

Winning two lotteries: When a 1 in 2.5 billion chance occurs more frequently than expected

So you want to be a millionaire? How about becoming an instant millionaire twice, in just two years?

A resident of Whistler, B.C. did just that by winning the $2.2 million grand prize in the B.C. Cancer Foundation Lifestyle Lottery just two years after winning the $1 million prize in the Surrey Memorial Hospital Lottery.

How lucky can you be—after all, just living in Whistler B.C. would be nirvana for many mere mortals. The chances of this "rare" event occurring was dutifully reported by the Vancouver Province, the Globe and Mail, and the CTV News as one in 2.5 billion. Is this correct?

The one in 2.5 billion figure was computed knowing that the winner had purchased two tickets out of the 100 000 available tickets for each of these two lotteries. Simple probability rules then give the chance of winning both lotteries as:

$$\textbf{Probability}\left(\begin{array}{c}\text{winning}\\ \text{both lotteries}\end{array}\right) = \frac{2}{100\ 000} \times \frac{2}{100\ 000},$$

which was the reported result.

However, this is misleading due to the common fallacy in probability computations of confusing the probability of a specific event with the probability of the general class of events. In this case, if you or I had won these two lotteries, or if the winner had won two different lotteries, the headlines would be identical.

As an analogy, the probability that you will be involved in two accidents on your way home from work or school today is very small, but the probability that someone, somewhere in North America will be involved in two accidents sometime in the year is quite large.

Computing the chances of these more general events requires additional information that is not easily collected, so some reasonable guesses must be made. Suppose that there are 10 000 people in B.C. who regularly buy two tickets in each of 25 lotteries each year with 100 000 tickets sold in each lottery. What is the chance of one person winning twice? To start, the chance of a particular person winning any pair of lotteries is 1/2.5 billion. But there are $\frac{25\times 24}{2} = 300$ possible pairs of lotteries that could be won by each of the 10 000 players, or roughly $3\ 000\ 000 = 300 \times 10\ 000$ opportunities for this to occur. A rough approximation then gives the overall probability of one the players winning any pair of lotteries as about $3\ 000\ 000/2\ 500\ 000\ 000 = 1.2/1000$, or about 0.1%. Somewhat unusual, but hardly overwhelming! Considering that across Canada there are similar situations every year in each of the provinces, the probability that it would happen to someone, somewhere in Canada, during a five-year window of opportunity is not very unusual at all!

This is an example of a class of problems called *occupancy problems*, one of which is the familiar birthday problem. A discussion of this and related problems can be found in the reference list at the end of this article.

The B.C. winner discussed above can't claim to be Canada's "luckiest winner." That honour goes to Maurice and Jeanette Garlepy of Alberta, who beat the approximately 14 million-to-one odds of winning the grand prize for Lotto 6/49 twice. Simple probability computations show the likelihood of this event as $(1/14\ 000\ 000)^2$, or about 1/200 trillion. What are the actual odds assuming 2 000 000 lottery players play biweekly for 10 years (1000 games)? What about all the other lotteries in Canada and the U.S.?

The correct odds are difficult to compute because of insufficient information. However, the correct odds must remain in the "human" realm. Lottery officials and newspaper editors, either because such events are not commonplace or because they like to calculate numbers with many zeroes (or perhaps because they think the public will buy more tickets) continue to make the odds of "interesting" events much more preposterous than they really are.

References:

[1] Diaconis, R. and Mosteller, F. Methods for Studying Coincidences. Journal of the American Statistical Association, 84, 853-861, 1989. Despite being in a scientific journal, this is a very readable article readily understood with high-school math knowledge.

[2] Hanley, J. A. Jumping to Coincidences: Defying the Odds in the Realm of the Preposterous. American Statistician, 46, 197-202, 1992. More examples of probability problems.

[3] Kolata, G. 1-in-Trillion Coincidence, You Say? Not Really, Experts Find. New York Times, February 27, 1990, c1. A newspaper account of the "incredible" two wins by the same person in an east coast lottery.

[4] Paulos, J.A. Innumeracy: Mathematical Illiteracy and Its Consequences. New York: Hill and Wang, 1988. A general introduction to the problems of innumeracy.

"ALTHOUGH IT DIDN'T HELP ANY ACTUAL PATIENT, OUR STATISTICIANS HAVE BEEN ABLE TO PROVE IT'S A VERY EFFECTIVE MEDICINE."

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Little Bits of Geometry

From Rabbits to Roses
A Geometric Mystery Story (continuation)
by Klaus Hoechsmann†

This is the promised continuation of our mystery series "**The Rose and the Nautilus**." The last time we met, we saw the connection of Fibonacci's down-to-earth sequence of "rabbit numbers":

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ 233,\ 377, \ldots,$$

with the elusive and legendary divine proportion, incarnated by rectangles with a strange property: when you cut a square away from one of them, you are left with a rectangle of the *same shape*—so you can continue cutting off squares indefinitely. So far, we don't know for sure that such rectangles exist—they may be a pipe dream or a **pie in the sky**. All we have our hands on, so far, are number sequences like the one above, where two successive numbers (high enough in the sequence) give us an *approximately* golden rectangle.

Today we'll make the journey from the golden rectangle to the golden triangle, the major building block of the pentagram (the five-cornered star on U.S. military vehicles), and hence the Pentagon (home of U.S. military brass). How fitting that the emblem of this remarkable institution shares its basic symmetry with the rose and many other flowers, from buttercups to petunias!

On the right side of Figure 1 you see a rose. The underlying pentagram (shown in light blue) is made up of golden triangles, each of which is obtained from a golden rectangle by collapsing one of the shorter sides to a point, like so:

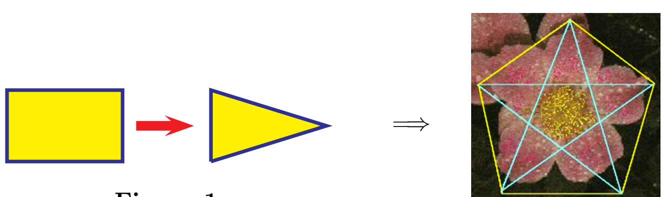

Figure 1

And if we do not have a perfect golden triangle handy, we'll take an approximate one made of rabbit numbers.

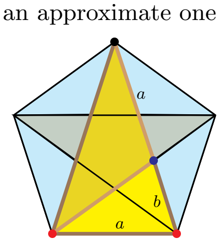

Figure 2

If you were given the large, two-tone yellowish triangle standing in Figure 2, you would not have much trouble filling in the rest of the diagram, would you? The proportions of this triangle are determined by $a = 144$ (base) and $a+b = 233$ (side) units of length—two successive rabbit numbers.

And the pentagram connection? If we knew that the base angle of the large two-tone triangle is equal to twice its vertex angle, the latter would be 1/5 of 180 degrees—see? Now, a/b is very nearly equal to $(a+b)/a$, which means that the lower pale yellow triangle is well-nigh isosceles, and the distance from the lower left red dot to the blue dot is nearly equal to a. Hence, the upper deep yellow triangle is well-nigh isosceles, and its exterior angle at the blue dot is roughly twice the vertex angle—Q.E.D.

Do I hear you grumbling? You are not happy with all this uncontrolled fuzziness? You'd like crisp, clean equality instead of all those "well-nighs"? Well, last year we tried to work with so-called "real" numbers, and you hated them when you were asked to produce crisp, clean proofs. And then somebody said that it all came down to Cauchy sequences and/or Dedekind cuts plus continuity, and when we tried that approach you all freaked out. Let's face it: if you want *numerical* proportions, you must choose between honest imperfection (e.g., rabbits) or dishonest perfection (e.g., root five). This is not rocket science: Eudoxos realized it way back around 400 B.C. So, let's get back to geometry, shall we? I admit that we have not yet constructed a perfect golden rectangle, but I promise you it isn't hard. For today, let us suppose it done and continue from there.

Our task is to repeat the argument pertaining to the pentagram connection (Figure 2) without any "nearly," "roughly," or "well nigh." Just go over it again, and you'll see that everything will be perfectly exact, as long as the pale yellow triangle in the lower right of Figure 2 is truly isosceles.

Into Geometry

We have *defined* what a golden rectangle should be, but never actually shown how to get one. Let's fill that gap now, using only a compass and a straight-edge (i.e., the notions of a circle and a straight line).

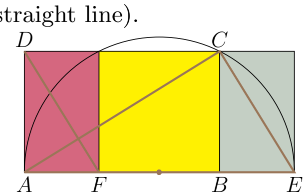

Figure 3

Starting with the yellow square (see Figure 3), we draw a circle centered at the midpoint of its base and going through the top corner C. It will meet the base line FB at two points A and E, and the triangle AEC has a right angle at C—okay? (If you doubt it, just cut the triangle by a line joining C to the center of the circle, and tally the angles in the two isosceles pieces you get.) Since the purple rectangle is just the gray one shifted over, its diagonal is also perpendicular to AC. If you turned it through 90 degrees, this diagonal would line up with AC. Thus, the rectangle $ABCD$ is similar to the purple one and is therefore golden. Any questions?

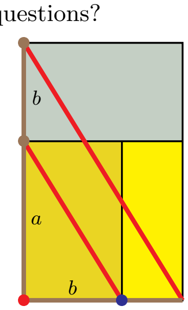

Figure 4

In Figure 4, we are looking at a golden rectangle. We remove the square at the bottom, shown in two shades of yellow. The residual rectangle at the top is gray, and a copy of it is shown (left side) inside the yellow square. This gray rectangle as well as its copy is similar to the original *big* rectangle, so the two red diagonals are parallel.

Imagine them as rubber bands, and the dot in the lower left corner as a hinge. Of course, we are *not* dealing with rubber bands and hinges, but with geometric

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entities in our minds. Yet this metaphor allows us to avoid tedious descriptions, and suggests that we speak of the brown vertical and horizontal sides as bars attached to that hinge. In Figure 5, see how we make a big isosceles triangle by simply swinging the vertical bar through a small angle. The *rubber bands will remain parallel.* That is important, because it means that the smaller triangle in the lower left has the same base angles as the big one. Do you agree? You are not quite convinced?

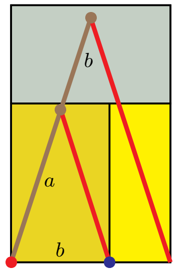


Figure 5

How perceptive of you—and right you are: mechanics is not geometry. I promise to come back to those rubber bands. But first let us see how to get a pentagram from the triangle in Figure 5, *supposing* that the parallelism holds up. The two-tone yellowish triangle in Figure 2 is, of course, our double triangle from Figure 5, with the smaller copy laid on its side. Does everything fit as it should?

First the angles: since big and small have the same base angles, small fits snugly into the lower right corner of big. Then the sides: the one labelled "a" in Figure 5 was the side of the square in Figure 4, hence fits exactly along the base. So it works! But remember: now that we have moved into geometry, "a" and "b" are line segments, not numbers.

Desargues

This might seem a bit lengthy, but if you mull it over, it's quite straight forward—except for the parallelism of those rubber bands, which seems so obvious mechanically, but hard to prove geometrically. In fact, it is false in pure plane geometry—some diligent geometers have found counterexamples—but becomes inevitably true if the plane sits in a three-dimensional space. This was discovered by Monsieur Girard Desargues about 350 years ago.

Here he is, on the left, staring at his theorem (its "affine" version), shown on the right in Figure 6. What does it look like to you? A leaning tower with two platforms in it? Good! What it says is this: if two triangles ABC and $A'B'C'$ are lined up so that the lines AA', BB', and CC' meet at a single point, and if AB is parallel to $A'B'$ and BC parallel to $B'C'$, it follows that AC is also parallel to $A'C'$. Obvious?

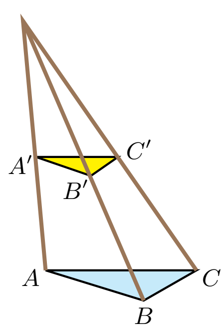


Figure 6

It almost *would* be if we were looking at a spatial setup. Then the hypothesis would quickly imply that the entire two "platforms" are parallel. But this is meant as a *flat, planar* diagram, and, as I have said, the theorem is impossible to prove within purely planar geometry.

You think you can prove it by using the equations of those lines? Goodness, now we are really turning around in circles! You want coordinates again—but not all planes *have* coordinates. Now that you've brought it up, let's get this straight: yes, you can prove the theorem by linear equations in any Cartesian plane, even if only rational coordinates are allowed, but the result is useless for golden triangles unless you allow *irrational* ones. But that's precisely what we wanted to *avoid* this semester, isn't it?

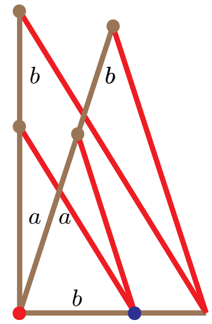


Figure 7

We have gone way over time—let's prove Girard's theorem at our next meeting. You will see that the proof is clever but not hard to follow. We pretend that his diagram is the planar projection of a spatial gizmo, then transfer the hypothesis to the latter, draw our conclusion up there in space and finally project down again. For today, let me just show you how it helps to keep our "rubber bands" parallel.

Figure 7 shows the bare bones of Figures 4 and 5 superimposed. Do you see the rubber bands? In their original position (attached to the vertical bar) they are parallel. In Figure 8, we added two short red lines—which are also parallel, being the bases of two isosceles triangles with the same vertex angle. Hence, the rubber bands remain parallel in their new position (attached to the slanted bar). *Et voilà!*

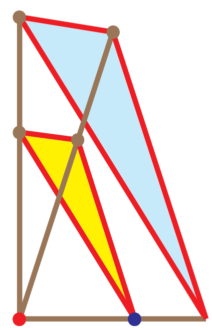
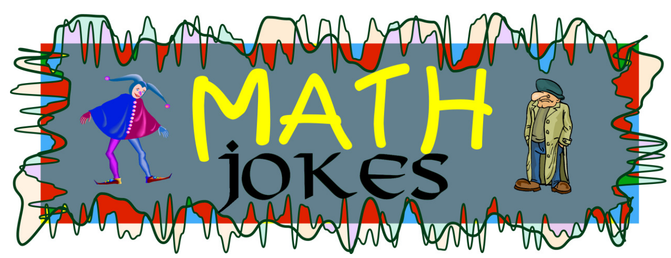


Figure 8

More about Girard Desargues? He was an engineer and mathematician who lived, worked, and died in Lyons, France. He did a lot of fine work and is known as the "father of projective geometry." That's the geometry of perspective. Please, now, let me go: I barely have time to get to my concert. More next time... yes, yes, I promise.



A mathematician and a stockbroker go to the races to bet on horses. The stockbroker suggests a bet of $10 000. That's too much for the mathematician's taste—first, he wants to understand the rules, have a look at the horse, etc.

"Don't worry," the stockbroker says. "I know an empirical algorithm that allows me to find the number of the winning horse with absolute certainty."

This does not convince the mathematician.

"You are too theoretical!" the stockbroker exclaims, and puts his $10 000 on a horse.

The horse comes in first—making the stockbroker even richer than he already is. The mathematician is baffled.

"What is your algorithm?" he wants to know.

"It's rather easy. I have two children, three and five years old. I add up their ages and bet on that number."

"But three plus five is eight—and that horse had number nine!"

"I told you that you're too theoretical! Didn't I just experimentally prove that my calculation is correct?!"

Polyhedra with Six Vertices

by Richard Ng†

In his June 1961 *Mathematical Games* column in *Scientific American*, later anthologized in [1], Martin Gardner posed nine problems. One of them was to count the number of different polyhedra with six faces. The solution given was that of John McClellan, who published his result in [2]. In this paper, we give an alternative and simpler solution to an equivalent problem, that of counting the number of different polyhedra with six vertices.

The left side of Figure 1 shows the skeleton of a tetrahedron. If we pull out the bottom face and compress the other onto it, we obtain the planar graph shown on the right side of Figure 1. A graph is planar if its edges only intersect at vertices.

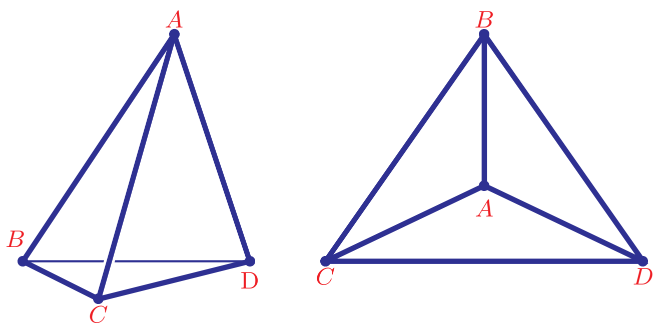


Figure 1
Polyhedron with Four Vertices

A non-planar graph cannot be the skeleton of a polyhedron. There are two basic non-planar graphs. The first one, called K_5, is shown on the left side of Figure 2. It consists of five vertices joined to one another. The second one, called $K_{3,3}$, is shown on the right side of Figure 2. Its six vertices are divided into two sets of equal size, and two vertices are joined by an edge if and only if they belong to different sets.

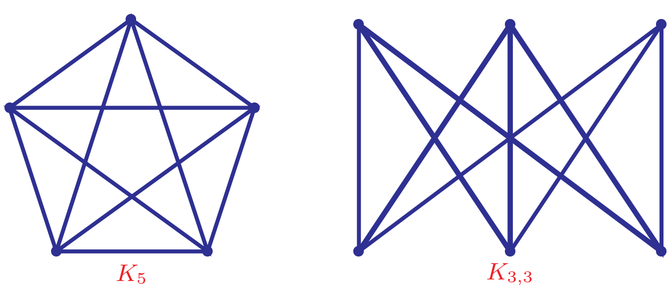


Figure 2
Non-Planar Graphs

† **Richard Ng** wrote this article while he was a grade 11 student at Archbishop MacDonald High School in Edmonton, in collaboration with Professor Andy Liu of the University of Alberta.

The number of edges meeting at a vertex is called the degree of the vertex. In a planar graph, which is the skeleton of a polyhedron, the degree of each vertex is at least 3. If there are only four vertices, then the degree of each vertex is at most 3. Hence, the degree of the four vertices must be $(3,3,3,3)$. Thus, the tetrahedron in Figure 1 is the only polyhedron with four vertices.

If the polyhedron has five vertices, the degree of its vertices may be $(3,3,3,3,3)$, $(3,3,3,4,4)$, $(3,4,4,4,4)$, $(3,3,3,3,4)$, $(3,3,4,4,4)$, or $(4,4,4,4,4)$. However, the first three sets are not feasible since the sum of all the degrees must be an even number, equal to twice the number of edges. The fourth is a square pyramid, shown on the left side of Figure 3; the fifth is a double triangular pyramid, shown on the right side of Figure 3. The last set is not the skeleton of a polyhedron because it is the non-planar graph K_5.

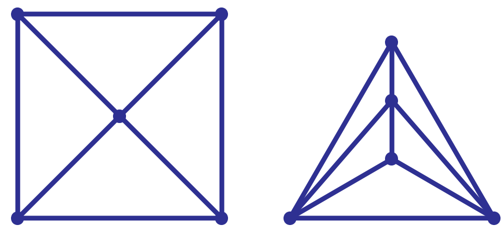


Figure 3
Polyhedra with Five Vertices

How many polyhedra with six vertices are there? The degree of each vertex is 3, 4, or 5. The following are all the possibilities where the sum of the degrees is even:

$(3,3,3,3,3,3)$, $(3,3,3,3,3,5)$, $(3,3,3,3,4,4)$, $(3,3,3,3,5,5)$,
$(3,3,3,4,4,5)$, $(3,3,4,4,4,4)$, $(3,3,3,5,5,5)$, $(3,3,4,4,5,5)$,
$(3,4,4,4,4,5)$, $(4,4,4,4,4,4)$, $(3,3,5,5,5,5)$, $(3,4,4,5,5,5)$,
$(4,4,4,4,5,5)$, $(3,5,5,5,5,5)$, $(4,4,5,5,5,5)$, $(5,5,5,5,5,5)$.

Among these 16 sets, four have two different forms each. So, there are altogether 20 cases to be considered. We discover that seven of them represent polyhedra. These are shown in Figures 4(a) and 4(b).

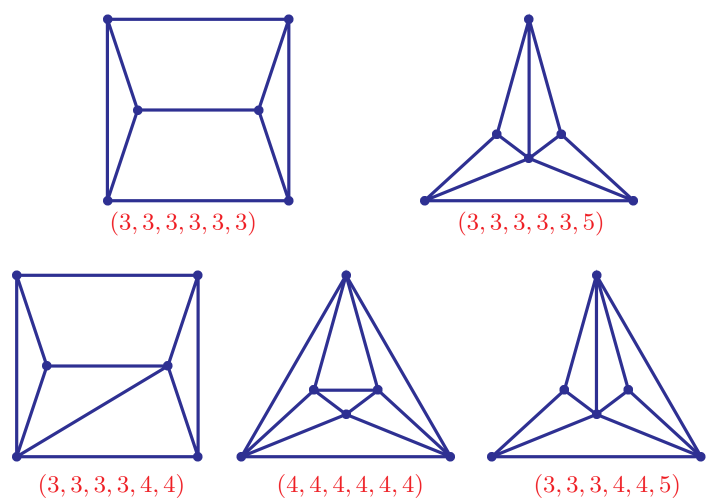
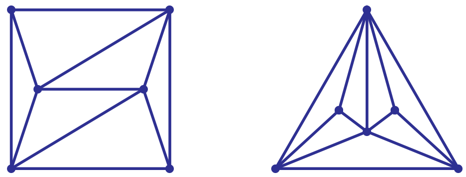


Figure 4(a)
Polyhedra with Six Vertices.



(3, 3, 4, 4, 4, 4) (3, 3, 4, 4, 5, 5)

Figure 4(b)
Polyhedra with Six Vertices

We conclude with the proof that there are only seven polyhedra with six vertices. Let us draw the graphs of the 20 cases identified above. Since most edges are present, we shall draw instead those that are missing. We quickly discover that the two cases (3, 3, 5, 5, 5, 5) and (3, 5, 5, 5, 5, 5) cannot be drawn, leaving only the eighteen cases in Figure 5.

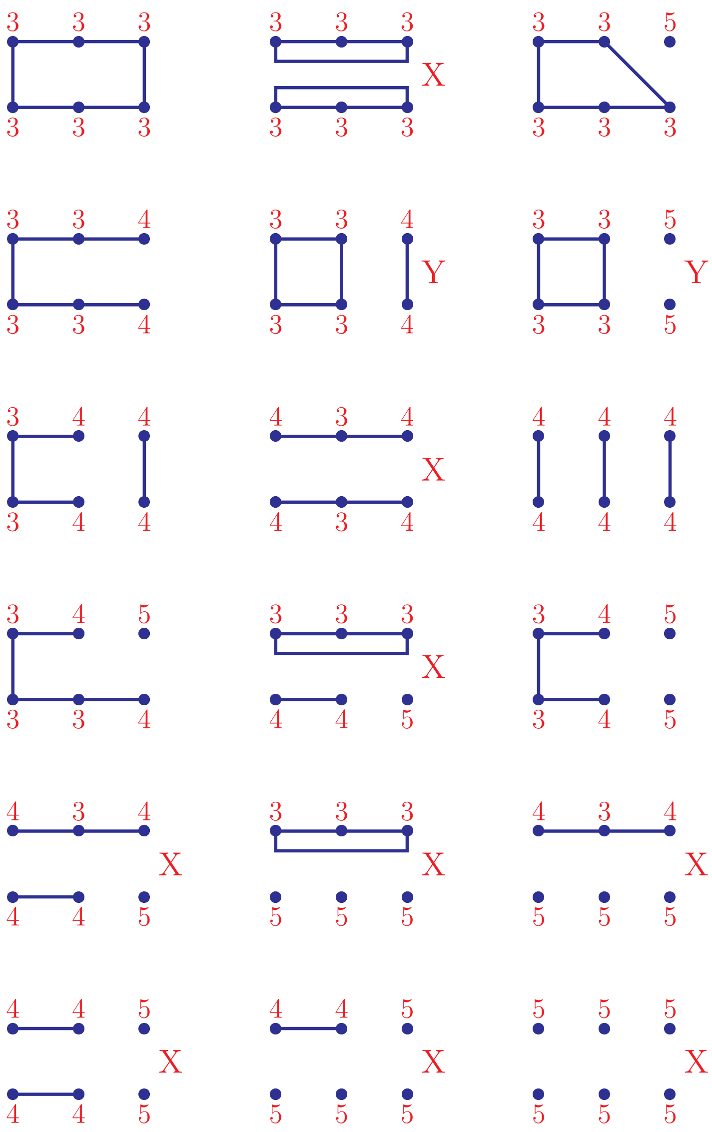


Figure 5
Case Studies

The seven unmarked cases in Figure 5 are the polyhedra in Figure 4. Those marked with an X are non-planar since they contain $K_{3,3}$. While the two cases with a Y are planar graphs, they are nevertheless not skeletons of polyhedra.

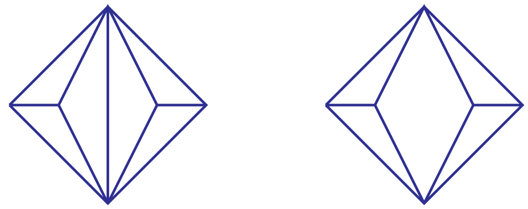


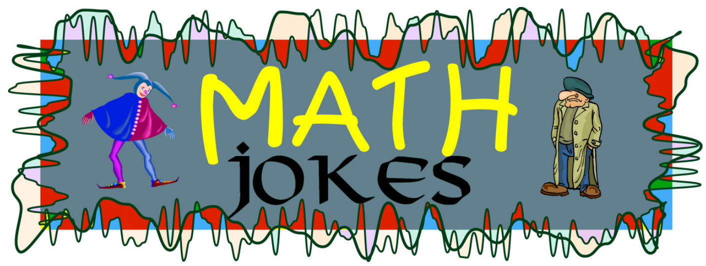
Figure 6
Impossible Polyhedra

Consider first the case (3, 3, 3, 3, 5, 5) on the left side of Figure 6. It consists of two tetrahedra joined along an edge. It is not the skeleton of a polyhedron. In the case (3, 3, 3, 3, 4, 4) on the right side of Figure 6, the two quadrilateral faces have two common vertices, those of degree 4, but they are adjacent. This is also not a skeleton of a polyhedron.

References:

[1] Martin Gardner. New Mathematical Diversions. Mathematical Association of America, 1995, pages 224–225 and 233.

[2] John McClellan. The Hexahedra Problem. Recreational Mathematics Magazine, volume 4, 1961, pages 34–40.

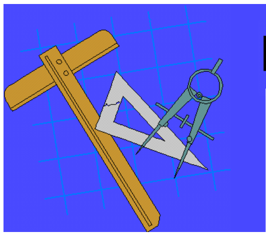


Four friends have been doing really well in their calculus class: they have received top grades for their homework and on the midterm. So when it's time for the final, they decide not to study on the weekend before, but to drive to another friend's birthday party in another city—even though the exam is scheduled for Monday morning. As it happens, they drink too much at the party, and on Monday morning, they are all hungover and oversleep. When they finally arrive on campus, the exam is already over.

They go to the professor's office and offer him an explanation, "We went to our friend's birthday party, and when we were driving back home very early on Monday morning, we suddenly had a flat tire. We had no spare, and since we were driving on back roads, it took hours until we got help."

The professor nods sympathetically and says, "I see that it was not your fault. I will allow you to make up for the missed exam tomorrow morning."

When they arrive early on Tuesday morning, the students are taken by the professor to a large lecture hall and are seated so far apart from each other that, even if they were to try, they would have no chance to cheat. The exam booklets are already in place, and confidently, the students start writing. The first question—five points out of 100—is a simple exercise in integration, and all four finish it within 10 minutes. The first to complete the problem turns over the page of the exam booklet and reads the next one: Problem 2 (95 points out of 100): *Which tire went flat?*



Math Strategies

Inequalities for Convex Functions (Part II)

by Dragos Hrimiuc†

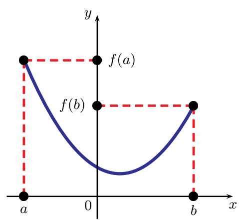
In the December 2001 issue of π *in the Sky*, we used convex functions to obtain several important inequalities. In this note we will describe another strategy to derive inequalities using convex functions.

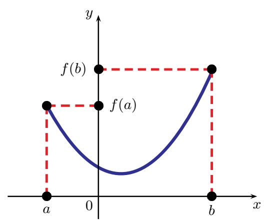
Inequalities with Constraints

If $f : [a,b] \to \mathbb{R}$ is a convex function, it achieves its maximum value at a or at b; that is,

$$f(x) \le \max\{f(a), f(b)\}, \text{ for every } x \in [a,b]. \qquad (1)$$

You can see this property demonstrated in the graphs below:





We can write a proof of (1). Indeed, $x \in [a,b]$ if and only if there exists $\lambda \in [0,1]$ such that $x = \lambda a + (1-\lambda)b$. If we set $M = \max\{f(a), f(b)\}$, we get:

$$\begin{aligned} f(x) &= f(\lambda a + (1-\lambda)b) \\ &\le \lambda f(a) + (1-\lambda) f(b) \\ &\le \lambda M + (1-\lambda)M = M. \end{aligned}$$

□

The inequality (1) can be extended for functions of n-variables. Assume that we have a function of n variables

$$F = F(x_1, \ldots, x_n), \quad x_i \in [a_i, b_i], \quad i = 1, \ldots, n.$$

We say that F is *convex in x_k* if

$$f(x_k) = F(x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n)$$

is convex for every $x_i \in [a_i, b_i]$, $i = 1, \ldots, n$, $i \ne k$. That is, by keeping $x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n$ fixed, we get a function of x_k only, which is a convex function.

Theorem: If F is convex in every $x_i \in [a_i, b_i]$, $i = 1, \ldots, n$, then

$$F(x_1, \ldots, x_n) \le \max_{\substack{t_i \in \{a_i, b_i\} \\ i = 1, \ldots, n}} F(t_1, \ldots, t_n). \qquad (2)$$

That is, the maximum value of F must be achieved at one of the 2^n vertices of the n-box $[a_1, b_1] \times \ldots \times [a_n, b_n]$.

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Proof: By using (1) and taking into account that F is convex in x_1, we get

$$F(x_1, \ldots, x_n) \le \max_{t_1 \in \{a_1, b_1\}} F(t_1, x_2, \ldots, x_n)$$

for every $x_i \in [a_i, b_i]$, $i = 1, \ldots, n$. On the other hand, F is convex with respect to x_2. Therefore,

$$F(t_1, x_2, \ldots, x_n) \le \max_{t_2 \in \{a_2, b_2\}} F(t_1, t_2, x_3, \ldots, x_n)$$

for every $t_1 \in \{a_1, b_1\}$, $x_i \in [a_i, b_i]$, $i = 2, \ldots, n$. From the last two inequalities we get

$$F(x_1, \ldots, x_n) \le \max_{\substack{t_i \in \{a_i, b_i\} \\ i = 1, 2}} F(t_1, t_2, x_3, \ldots, x_n)$$

for every $x_i \in [a_i, b_i]$, $i = 1, \ldots, n$. Repeating the above argument, we easily get (2). □

Example 1. (Kantorovich's Inequality)
If $p_i \ge 0$ and $0 < a \le x_i \le b$, $i = 1, \ldots, n$, then

$$\begin{aligned} \left(\sum_{i=1}^n p_i x_i\right)\left(\sum_{i=1}^n \frac{p_i}{x_i}\right) &\le \frac{(a+b)^2}{4ab}\left(\sum_{i=1}^n p_i\right)^2 \\ &\quad - \frac{(a-b)^2}{4ab} \min_{\substack{X \cup Y = A_n \\ X \cap Y = \emptyset}} \left(\sum_{i \in X} p_i - \sum_{j \in Y} p_j\right)^2, \end{aligned} \qquad (3)$$

where $A_n = \{1, 2, \ldots, n\}$.

Solution: Consider the function

$$F(x_1, \ldots, x_n) = \left(\sum_{i=1}^n p_i x_i\right)\left(\sum_{i=1}^n \frac{p_i}{x_i}\right).$$

Setting $M_i = \sum_{\substack{j=1 \\ j \ne i}}^n p_j x_j$ and $N_i = \sum_{\substack{j=1 \\ j \ne i}}^n \frac{p_j}{x_j}$, we see that

$$\begin{aligned} f(x_i) &= (p_i x_i + M_i)\left(\frac{p_i}{x_i} + N_i\right) \\ &= p_i^2 + M_i N_i + p_i N_i x_i + p_i M_i \frac{1}{x_i}. \end{aligned}$$

Now f is convex, because $f''(x_i) > 0$; hence F is convex in x_i, $i = 1, \ldots, n$.

On using (2), we see that F obtains its maximal value at one of the 2^n vertices of the n-box $[a_1, b_1] \times \ldots \times [a_n, b_n]$.

Therefore,

$$F(x_1 \ldots, x_n) \le \max_{\substack{X \cup Y = A_n \\ X \cap Y = \emptyset}} \left(a \sum_{i \in X} p_i + b \sum_{j \in Y} p_j\right)\left(\frac{1}{a}\sum_{i \in X} p_i + \frac{1}{b}\sum_{j \in Y} p_j\right). \qquad (4)$$

Taking into account the identity

$$AB = \frac{1}{4ab}(A + abB)^2 - \frac{1}{4ab}(A - abB)^2,$$

we get

$$\begin{aligned} &\left(a \sum_{i \in X} p_i + b \sum_{j \in Y} p_j\right)\left(\frac{1}{a}\sum_{i \in X} p_i + \frac{1}{b}\sum_{j \in Y} p_j\right) \\ &\quad = \frac{(a+b)^2}{4ab}\left(\sum_{i=1}^n p_i\right)^2 - \frac{(a-b)^2}{4ab}\left(\sum_{i \in X} p_i - \sum_{j \in Y} p_j\right)^2. \end{aligned}$$

On using this identity in (4), we then get (3). $\square$

Example 2. If $0 < a \le x_i \le b$, $i = 1, \ldots, n$, then

$$\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} \frac{1}{x_i}\right) \le \frac{(a+b)^2}{4ab}n^2 - \frac{(a-b)^2}{4ab}\left[\frac{1-(-1)^n}{2}\right].$$

Solution: Use Example 1. $\square$

Example 3. If α, β, $\gamma \in [0, \pi/2]$, prove that

$$\left(\cos\frac{\alpha}{2} + \cos\frac{\beta}{2} + \cos\frac{\gamma}{2}\right)\left(\sec\frac{\alpha}{2} + \sec\frac{\beta}{2} + \sec\frac{\gamma}{2}\right) \le 5 + 3\sqrt{2}.$$

Solution: Since α, β, $\gamma \in [0, \pi/2]$, we have $\frac{\alpha}{2}$, $\frac{\beta}{2}$, $\frac{\gamma}{2} \in [0, \frac{\pi}{4}]$. Hence $\cos\frac{\alpha}{2}$, $\cos\frac{\beta}{2}$, $\cos\frac{\gamma}{2} \in [\sqrt{2}/2, 1]$. Now we can use Example 2. $\square$

Example 4. If a, b, $c \in [0, 1]$, then

$$\frac{a}{b+c+1} + \frac{b}{c+a+1} + \frac{c}{a+b+1} + (1-a)(1-b)(1-c) \le 1.$$

(USA Mathematical Olympiad, 1980)

Solution: The function

$$F(a,b,c) = \frac{a}{b+c+1} + \frac{b}{c+a+1} + \frac{c}{a+b+1} + (1-a)(1-b)(1-c)$$

is convex in each a, b, and c. Hence, its maximum value is achieved at one of the 2^3 vertices of the box $[0,1]\times[0,1]\times[0,1]$; that is, at one of the following points: $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,0)$, $(0,1,1)$, $(1,0,1)$, $(1,1,1)$. We deduce that $\max\limits_{a,b,c\in[0,1]} F(a,b,c) = 1$; this value is achieved at each of the vertices of the box. $\square$

Example 5. If $0 < a \le x_i \le b$, $i = 1, \ldots, n$, then

$$\left(\sum_{i=1}^{n} x_i\right)^2 \ge \frac{4nab}{(a+b)^2}\sum_{i=1}^{n} x_i^2.$$

Solution: By taking $p_i = x_i$ in (3), we get

$$\left(\sum_{i=1}^{n} x_i^2\right)\cdot n \le \frac{(a+b)^2}{4ab}\left(\sum_{i=1}^{n} x_i\right)^2,$$

which is the required inequality. $\square$

By using the method described in this note, prove the following inequalities:

Problem 1. If a, b, $c \in [0, 1]$, then

$$\frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+c+a} + k(1-abc) \le \max\{1, 3+k\}.$$

Problem 2. If a, b, $c \in [1, 2]$, then

$$(a+5b+9c)\left(\frac{1}{a} + \frac{5}{b} + \frac{9}{c}\right) \le 225.$$

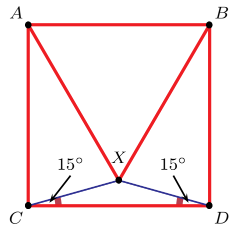
When does equality occur?

Solution to a Geometry Problem

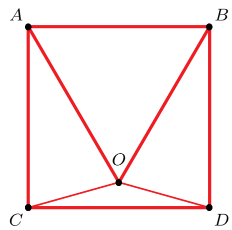
by

Brendan Capel[1] and **Alan Tsay**[2]

On October 23, 2001, five University of Alberta mathematicians invaded the grade 10 and 11 classes at Tempo School in Edmonton. See [2] for a brief account. The unidentified mathematician was Henry van Roessel, who was behind the camera.

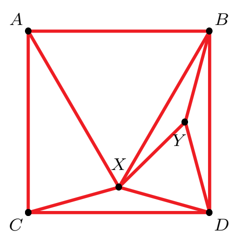


Dragos Hrimiuc offered a \$5 prize to anyone who could solve the following geometry problem: "Consider a square $ABCD$ with a point X inside it such that $\angle XCD = \angle XDC = 15°$. Prove that triangle ABX is equilateral." The prize was not claimed that day.



Later, we found the same problem in [1]. Here is the published solution: Let O be the point inside $ABCD$ such that triangle ABO is equilateral. Then $\angle OAB = 60°$, so that $\angle CAO = 30°$. Now $AO = AB = AC$. Hence $\angle ACO = \angle AOC = \frac{1}{2}(180° - \angle CAO) = 75°$, so that $\angle OCD = 15°$. Similarly, $\angle ODC = 15°$. It follows that the point O must coincide with the point X. Since ABO is an equilateral triangle, so is ABX.

We wondered if there is a direct approach that will solve this problem. After some work, we came up with the following argument.



Let Y be the point inside triangle DBX such that DXY is equilateral. Then $\angle XDY = 60°$, so that $\angle BDY = 15° = \angle XDC$. Since $DY = DX$ and $DB = DC$, triangles DBY and DCX are congruent to each other, so that

$$\angle BYD = \angle CXD = 180° - \angle XCD - \angle XDC = 150°$$

and $BY = CX = DX = XY$. Now

$$\angle BYX = 360° - \angle BYD - \angle DYX = 150°,$$

so that

$$\angle BXY = \angle XBY = \frac{1}{2}(180° - \angle BYX) = 15°.$$

It follows that $\angle BDX = \angle BXD$, so that $BX = BD$. By symmetry, $BX = AX$. Hence, ABX is indeed an equilateral triangle.

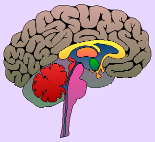
References:

[1] D. V. Fomin. Criminal Geometry—A Matter of Principle. Quantum, September/October 1991, 47–49 and 61.

[2] D. Stanley. π *in the Sky* at Tempo School, π *in the Sky*, December 2001, 28.

[1] **Brendan Capel** submitted this solution while he was a grade 10 student at Tempo School, Edmonton.

[2] **Alan Tsay** submitted this solution while he was a grade 9 student at Vernon Barford School, Edmonton.



Math Challenges

Problem 1. Let $f:\mathbb{R}\to\mathbb{R}$, $f(x)=x^2-5|x|+9$. Find all finite subsets $A\subset\mathbb{R}$ such that for every $x\in A$, $f(x)\in A$.

Problem 2. Find all positive integers m and n such that $1+2^n=3^m$.

Problem 3. Prove that each rational number $\frac{m}{n}$, with $0<\frac{m}{n}<1$, can be written as
$$\frac{m}{n}=\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_k},$$
where $0<a_1<a_2<\ldots<a_k$ are integers and each a_{r-1} is a divisor of a_r, for $r=2,3,\ldots,k$.

Problem 4. Inside of the square $ABCD$, take any point P. Prove that the perpendiculars from A on BP, from B on CP, from C on DP, and from D on AP are concurrent (i.e. they meet at one point).

Problem 5. A rectangular piece of cardboard is cut straight into two pieces. One of these two pieces is cut again in two, and so on. Find the minimum number of cuts that must be done such that among all the pieces there will be 2002 polygons with 2003 sides.

Problem 6. Let $f,\,g:[a,b]\to[0,\infty)$ be two convex functions, with $x_i\in[a,b]$, $p_i\ge 0$, $i=1,\ldots,n$. Show that for every $k>0$,
$$\left(\sum_{i=1}^{n}p_if(x_i)\right)\left(\sum_{i=1}^{n}p_ig(x_i)\right)\le\frac{1}{4k}M^2\left(\sum_{i=1}^{n}p_i\right)^2,$$
where $M=\max\{f(a)+kg(a),f(b)+kg(b)\}$.

Send your solutions to π *in the Sky*: Math Challenges.

Solutions to the Problems Published in the December, 2001 Issue of π *in the Sky*:

Problem 1. (By Edward T.H. Wang from Waterloo)
Let $A=\frac{a}{b+c-a}+\frac{b}{c+a-b}+\frac{c}{a+b-c}$ and $s=\frac{1}{2}(a+b+c)$ denote the semi-perimeter of the triangle. Then
$$\begin{aligned}A&=\tfrac{1}{2}\left(\tfrac{a}{s-a}+\tfrac{b}{s-b}+\tfrac{c}{s-c}\right)=\tfrac{1}{2}\left(\tfrac{s-(s-a)}{s-a}+\tfrac{s-(s-b)}{s-b}+\tfrac{s-(s-c)}{s-c}\right)\\&=\tfrac{1}{2}\left(\tfrac{s}{s-a}+\tfrac{s}{s-b}+\tfrac{s}{s-c}-3\right).\end{aligned}\tag{1}$$

Since $s-a>0$, $s-b>0$, and $s-c>0$, we have by the Cauchy–Schwarz Inequality that
$$\begin{aligned}\tfrac{s}{s-a}+\tfrac{s}{s-b}+\tfrac{s}{s-c}&=[(s-a)+(s-b)+(s-c)]\left(\tfrac{1}{s-a}+\tfrac{1}{s-b}+\tfrac{1}{s-c}\right)\\&\ge 9.\end{aligned}$$
On substituting the last inequality into (1), $A\ge 3$ follows immediately.

Alternative Solution to Problem 1: Let $f(x)=\frac{x}{s-x}$ for $x<s$. Since $f''(x)=2s/(s-x)^3>0$, we see that f is strictly convex. By Jensen's Inequality we get
$$\frac{a}{s-a}+\frac{b}{s-b}+\frac{c}{s-c}\ge 6\iff A\ge 3.$$
The inequality holds if and only if $a=b=c$.

Problem 2. (i) Let $f(x)=\tan^p x$, with $x\in(0,\frac{\pi}{2})$, $p\ge 1$. Since $f''(x)>0$, f is strictly convex on $(0,\frac{\pi}{2})$ and the required inequality follows from Jensen's Inequality (see 4, Math Strategies, π *in the Sky*, December 2001).

(ii) Let $f(x)=\ln(\sin x)$, with $x\in(0,\pi)$. Since $f''(x)=-\csc^2 x$, f is strictly concave on $(0,\pi)$. Using Jensen's Inequality (see 4, Math Strategies, π *in the Sky*, December 2001), we get
$$\ln\sin\alpha'+\ln\sin\beta'+\ln\sin\gamma'\le 3\ln\sin\frac{\alpha'+\beta'+\gamma'}{3},$$
where α', β', γ' are the angles of a triangle. This inequality can also be written as:
$$\sin\alpha'\sin\beta'\sin\gamma'\le\left(\frac{\sqrt{3}}{2}\right)^3.$$
Let $\alpha'=\frac{\pi}{2}-\frac{\alpha}{2}$, $\beta'=\frac{\pi}{2}-\frac{\beta}{2}$, $\gamma'=\frac{\pi}{2}-\frac{\gamma}{2}$. Now
$$\sin\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)\sin\left(\frac{\pi}{2}-\frac{\beta}{2}\right)\sin\left(\frac{\pi}{2}-\frac{\gamma}{2}\right)\le\frac{3\sqrt{3}}{8}.$$
That is,
$$\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}\le\frac{3\sqrt{3}}{8}.$$
Note: There was a typographical error in the original published statement of Problem 2(ii). The inequality should read as above. Note that the assumption that α, β, γ are the angles of an acute triangle is not necessary. Also, in Problem 3, a should be replaced by s.

Problem 3. (By Edward T.H. Wang from Waterloo)
By the Cauchy–Schwarz Inequality, we have
$$\begin{aligned}\sum_{k=1}^{n}\frac{s}{s-a_k}&=\frac{(n-1)s}{n-1}\sum_{k=1}^{n}\frac{1}{s-a_k}\\&=\frac{1}{n-1}\left[\sum_{k=1}^{n}(s-a_k)\right]\left[\sum_{k=1}^{n}\frac{1}{s-a_k}\right]\\&\ge\frac{n^2}{n-1}.\end{aligned}$$

Alternative Solution to Problem 3: Let $f(x)=\frac{s}{s-x}$, $x<s$. Since $f''(x)>0$ on $(-\infty,s)$, we see that f is strictly convex. Using Jensen's Inequality, we get
$$\frac{1}{n}\sum_{k=1}^{n}\frac{s}{s-a_k}\ge\frac{s}{s-\frac{s}{n}}\iff\sum_{k=1}^{n}\frac{s}{s-a_k}\ge\frac{n^2}{n-1}.$$

Problem 4.
Let $f(x)=\frac{x}{s+(m-1)x}$, where $s+(m-1)x>0$. Since
$$f''(x)=\frac{2s(1-m)}{[s+(m-1)x]^3},$$
we see that f is convex if $m\in[0,1]$ and is concave if $m\in[1,\infty)$.
Let $m\in[0,1]$. By Jensen's Inequality,
$$\sum_{k=1}^{n}\frac{a_k}{s+(m-1)a_k}\ge n\frac{\frac{a_1+\ldots+a_n}{n}}{s+(m-1)\frac{a_1+\ldots+a_n}{n}}.$$
Taking $s=a_1+\ldots+a_n$, we have $s+(m-1)a_k>0$ for $k=1,\,\ldots,\,n$. Hence
$$\frac{a_1}{ma_1+a_2+\ldots+a_n}+\ldots+\frac{a_n}{a_1+a_2+\ldots+ma_n}\ge\frac{n}{m+n-1},$$
as requested.
If $m\in[1,\infty)$, the inequality reverses since f is concave.

Problem 5. (By Edward T.H. Wang from Waterloo)
Let $\lambda_k=\frac{a_k}{a_1+a_2+\ldots+a_n}$, $k=1,2,\ldots,n$. Then $\lambda_k>0$ and $\sum_{k=1}^{n}\lambda_k=1$. Then, by the Weighted AM–GM Inequality (see 5, Math Strategies, π *in the Sky*, December 2001), we have
$$\begin{aligned}\frac{1}{a_1+a_2+\ldots+a_n}\sum_{k=1}^{n}a_kx_k^{1/a_k}&=\sum_{k=1}^{n}\lambda_kx_k^{1/a_k}\ge\prod_{k=1}^{n}x_k^{\frac{\lambda_k}{a_k}}\\&=\prod_{k=1}^{n}x_k^{\frac{1}{a_1+\ldots+a_n}}=\left(\prod_{k=1}^{n}x_k\right)^{\frac{1}{a_1+\ldots+a_n}},\end{aligned}$$
from which the requested inequality follows.

Problem 6.
Using the Hölder Inequality, we find
$$a(\sin x)^{\frac{1}{p}}+b(\cos x)^{\frac{1}{p}}\le(a^s+b^s)^{\frac{1}{s}}\left(\sin^{\frac{\ell}{p}}x+\cos^{\frac{\ell}{p}}x\right)^{\frac{1}{\ell}},$$

where $\frac{1}{\ell}+\frac{1}{s}=1$ and $\ell, s>1$. By choosing $\ell=2p$, we get $s=\frac{2p}{2p-1}$, and the required inequality follows immediately.

Problem 7. (By Y. Chen and Edward T.H. Wang from Waterloo)

We shall prove a stronger result:

$$|\sin a_1 \, \sin a_2 \, \dots \, \sin a_n| + |\cos a_1 \, \cos a_2 \, \dots \, \cos a_n| \le 1 \qquad (5)$$

for $n>2$ and for all $a_i \in \mathbb{R}$, where $i=1,\dots,n$. Clearly (5) implies the required inequality. We prove (5) by induction on n. For $n=2$, we have, by the Cauchy–Schwarz Inequality, that

$$|\sin a_1 \, \sin a_2| + |\cos a_1 \, \cos a_2| \le \sqrt{\sin^2 a_1 + \cos^2 a_1}\sqrt{\sin^2 a_2 + \cos^2 a_2} = 1.$$

That is, (5) holds for $n=2$. Suppose that (5) holds for $n=k\ge 2$. Then, for $a_{k+1}\in\mathbb{R}$, we have, by the Cauchy–Schwarz Inequality and the induction hypothesis, that

$$\begin{aligned} &|\sin a_1 \dots \sin a_k \, \sin a_{k+1}| + |\cos a_1 \dots \cos a_k \, \cos a_{k+1}| \\ \le\ & \sqrt{|\sin a_1 \dots \sin a_k|^2 + |\cos a_1 \dots \cos a_k|^2}\sqrt{\sin^2 a_{k+1}+\cos^2 a_{k+1}} \\ \le\ & |\sin a_1 \dots \sin a_k| + |\cos a_1 \dots \cos a_k| \le 1. \end{aligned}$$

Hence (5) holds for $n=k+1$, and this completes the induction.

Problem 8. If $x_1, x_2, \dots, x_n$ are positive real numbers and $x_1 \le x_2 \le \dots \le x_n$, then $x_1^n \le x_2^n \le \dots \le x_n^n$. By using Chebyshev's Inequality (see Math Strategies, π *in the Sky*, December 2000), we get

$$\sum_{\substack{i=1\\ i\neq k}}^{n} x_i^n = \sum_{\substack{i=1\\ i\neq k}}^{n} x_i^{n-1}x_i \ge \frac{1}{n-1}\left(\sum_{\substack{i=1\\ i\neq k}}^{n} x_i\right)\left(\sum_{\substack{i=1\\ i\neq k}}^{n} x_i^{n-1}\right),$$

for $k=1,2,\dots,n$. The AM–GM Inequality (see 5, Math Strategies, π *in the Sky*, December 2001) yields

$$\sum_{\substack{i=1\\ i\neq k}}^{n} x_i^{n-1} \ge (n-1)x_1\dots x_{k-1}x_{k+1}\dots x_n, \qquad k=1,2,\dots,n.$$

Therefore,

$$\sum_{\substack{i=1\\ i\neq k}}^{n} x_i^{n} \ge (x_1\dots x_{k-1}x_{k+1}\dots x_n)\sum_{\substack{i=1\\ i\neq k}}^{n} x_i, \qquad k=1,2,\dots,n.$$

This inequality can also be written as

$$\begin{aligned} & x_1^n+\dots+x_{k-1}^n+x_{k+1}^n+\dots x_n^n+x_1x_2\dots x_n \\ \ge\ & x_1\dots x_{k-1}x_{k+1}\dots x_n(x_1+x_2+\dots+x_n), \end{aligned}$$

or

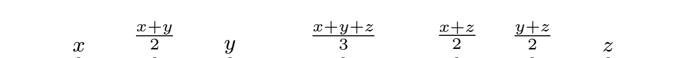
$$\begin{aligned} & \frac{1}{x_1x_2\dots x_n+x_1^n+\dots+x_{k-1}^n+x_{k+1}^n+\dots x_n^n} \\ \le\ & \frac{1}{x_1+\dots+x_n}\cdot\frac{1}{x_1\dots x_{k-1}x_{k+1}\dots x_n} \end{aligned}$$

for $k=1,2,\dots,n$. Adding up these inequalities, we get

$$\sum_{k=1}^{n}\frac{1}{x_1x_2\dots x_n+x_1^n+\dots+x_{k-1}^n+x_{k+1}^n+\dots x_n^n} \le \frac{1}{x_1x_2\dots x_n}.$$

On setting $x_k^n=a_k$ for $k=1,2,\dots,n$ we obtain the required inequality.

Problem 9. We may assume that $x\le y<z$.
Case 1: $y\le\frac{x+z}{2}$. Then we can draw the following picture:



We have

$$\frac{x+z}{2}=(1-\alpha)\frac{x+y+z}{3}+\alpha z, \qquad \alpha\in[0,1]$$

and

$$\frac{y+z}{2}=(1-\beta)\frac{x+y+z}{3}+\beta z, \qquad \beta\in[0,1].$$

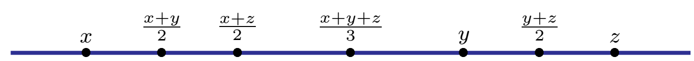
On solving the above equations for α and β, we find

$$\alpha=\frac{x+z-2y}{2(2z-x-y)} \quad\text{and}\quad \beta=\frac{y+z-2x}{2(2z-x-y)};$$

hence $\alpha+\beta=\frac{1}{2}$. Since f is convex, we have

$$\begin{aligned} f\left(\frac{x+z}{2}\right) &\le (1-\alpha)f\left(\frac{x+y+z}{3}\right)+\alpha f(z), \\ f\left(\frac{y+z}{2}\right) &\le (1-\beta)f\left(\frac{x+y+z}{3}\right)+\beta f(z), \\ f\left(\frac{x+y}{2}\right) &\le \frac{1}{2}f(x)+\frac{1}{2}f(y). \end{aligned}$$

Adding these three inequalities, we obtain the required inequality.
Case 2: $y>\frac{x+z}{2}$. The above picture becomes



Now, we can write:

$$\begin{aligned} \frac{x+y}{2} &= (1-\gamma)x+\gamma\tfrac{x+y+z}{3}, \qquad \gamma\in[0,1], \\ \frac{x+z}{2} &= (1-\delta)x+\delta\tfrac{x+y+z}{3}, \qquad \delta\in[0,1]. \end{aligned}$$

We can solve for γ and δ and get

$$\gamma=\frac{3}{2}\,\frac{y-x}{y+z-2x}, \quad\text{and}\quad \delta=\frac{3}{2}\left(\frac{z-x}{y+z-2x}\right),$$

hence $\gamma+\delta=\frac{3}{2}$. Again, since f is convex we have

$$\begin{aligned} f\left(\frac{x+y}{2}\right) &\le (1-\gamma)f(x)+\gamma f\left(\frac{x+y+z}{3}\right), \\ f\left(\frac{x+z}{2}\right) &\le (1-\delta)f(x)+\delta f\left(\frac{x+y+z}{3}\right), \\ f\left(\frac{y+z}{2}\right) &\le \frac{1}{2}f(y)+\frac{1}{2}f(z), \end{aligned}$$

and the required inequality follows by adding these inequalities.

Explanation of the joke *"Why do mathematicians often confuse Christmas and Halloween? Because Oct 31 = Dec 25."*

Notice that 31 in base 8 (Oct) = 25 in the decimal system (Dec). Solutions were received from: John Freal, Andrew Walenstein, Cindy Maldonado, Keith Edwards, Connye LaCombe, Chad Goobie, and John Boyer.

From our Readers

Dear Pi,

I recently picked up a copy of π *in the Sky* while visiting the Faculty of Science at the U of A. Fun reading!

One of your jokes asked how to split 14 cubes of sugar among three cups of coffee such that each receives an odd number of cubes. There actually is a way to do this: put 3 in the first cup, 3 in the second cup (or 1 and 5), then use the remaining 8 to form a cube of side length 2 and place the resulting cube in the third cup.

Cheers, John Boyer (Edmonton)

Cover Page Story

by George Peschke[†]

Oops!!! Just What Happened to Prof. Zmodtwo?

We don't really know. There is no limit to speculation though, with suggested answers ranging from: "He thought the interior of the jar was pleasantly warmer than his study," to "He was trying to escape a bunch of Greek characters—$\varrho\varpi\Upsilon\Theta$—that had suddenly appeared on his desk."

However, there have been some episodes in Prof. Binarius Zmodtwo's life that suggest a tragic error in computing with signs led to his unfortunate and very unexpected whereabouts.

We know that in high school young Binarius suffered from a severe form of Quadratic Formula Syndrome (QFS). Probably most of us can remember this experience:

$$(a+b)^2 = a^2 + b^2 \underline{+ 2ab},$$

the red ink coming from our teacher's corrective marker. With a stern face, our teacher might have proceeded to remind us

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b)\\ &= a(a+b)+b(a+b)\\ &= a^2+ab+ab+b^2\\ &= a^2+2ab+b^2.\end{aligned}$$

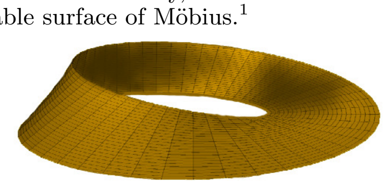
Thanks to our teacher's patience, most of us got it right eventually. Not so, however, Binarius. To him, this "$2ab$" was no less than an insult to what seems intuitively right: if you want to square a bracket, you square the things inside of it. So he just continued to insist on the incorrect formula $(a+b)^2 = a^2 + b^2$, which is a characteristic symptom of QFS.

He looked hard for a way to defend his perception, and eventually found it by adopting a simplified number system—one that contains only the numbers 0 and 1, and in which one computes $1 + 1 = 0$, and so $1 = -1$. Therefore,

$$1ab + 1ab = 0ab = 0,$$

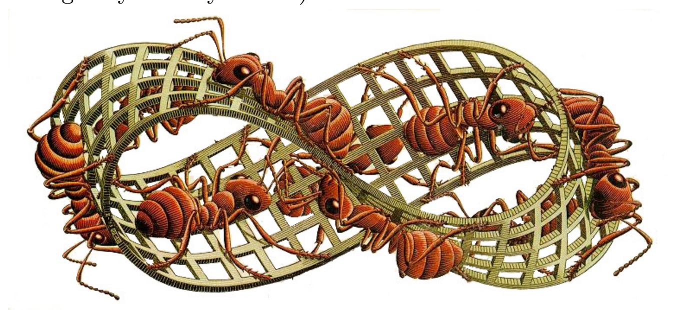
which justifies his version of the quadratic formula.

Binarius was so delighted with his discovery that he wrote extensively about it. He even dedicated an entire book on arithmetic to the subject. More recently, at an arts exhibition, he encountered the remarkable surface of Möbius.[1]



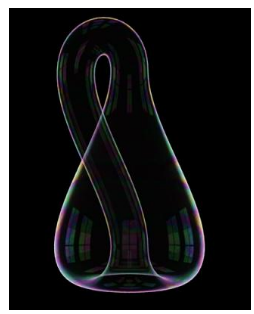
You can construct this surface by gluing the edges of a paper strip to form a loop with a twist.

An ant travelling on the Möbius surface would have an unexpected experience. Follow the uppermost ant once around the loop; it will finish on the side opposite to its starting point. After another round trip, it will arrive back at its original starting point. That is, each round trip causes a side swap, and the side swaps caused by two round trips have cancelled each other—cancelled, as if we had done nothing at all. Just like $1 + 1 = 0$! (This is amazing—try it out yourself!)



M.C. Escher's "Möbius Strip II" ©2002 Cordon Art B.V. - Baarn - Holland. All rights reserved.

Binarius Zmodtwo was fascinated. This surface was, in a way, just like his simplified number system. Then he encountered the bottle of Felix Klein.[2] Although we cannot construct the Klein bottle in our three-dimensional world (we can only create 3D projections, such as those available at http://www.kleinbottle.com), we can imagine it being built by gluing the opposite edges of the following paper strip, with like-coloured arrows aligned:



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In any event, Binarius Zmodtwo was thrilled to discover that the Klein bottle has no interior: if you wander along what appears to be the interior wall, you end up on what appears to be the exterior. Keep wandering, and you are back to where you started. Just like $1 + 1 = 0$ again!

When Mom's sweet homemade cherries arrived, perhaps Prof. Zmodtwo lacked a tool to open the jar and tried to use his theories instead!

Comment: These one-sided surfaces have another intriguing feature. Imagine if the ant in Escher's drawing was perfectly flat, so that it actually lived inside the surface. Such an ant could make a round trip and return as its own mirrored image! Try it: make a Möbius strip and two ants out of a sheet of clear plastic. On each ant mark its heart on the left side. Then, glue one ant to the surface and slide the other once around the strip. If before the trip both ants had matching heart positions, the returning ant will have its heart in the mirrored position.

The universe in which we live is somewhat like a surface. Only we have, in addition to the directions "forward–backward" and 'left–right," also the direction "up–down." The situation inside our universe is just like that of the flat ant living within the Möbius surface. So we ask: Is our universe like a one-sided surface, in which a trip around it would bring us back to the initial point as our mirrored image? As of yet, the answer to this question is unknown.

[†] **George Peschke** is a professor in the Department of Mathematical and Statistical Sciences at the University of Alberta. You can visit his web page at: http://www.ualberta.ca/dept/math/gauss/george/.

[1] August Ferdinand Möbius (1790–1868)—German mathematician and theoretical astronomer who is best known for his work in analytic geometry and in topology.

[2] Christian Felix Klein (1849–1925)—German mathematician whose unified view of geometry as the study of the properties of a space that are invariant under a given group of transformations, known as the Erlanger Program, profoundly influenced mathematical developments.

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