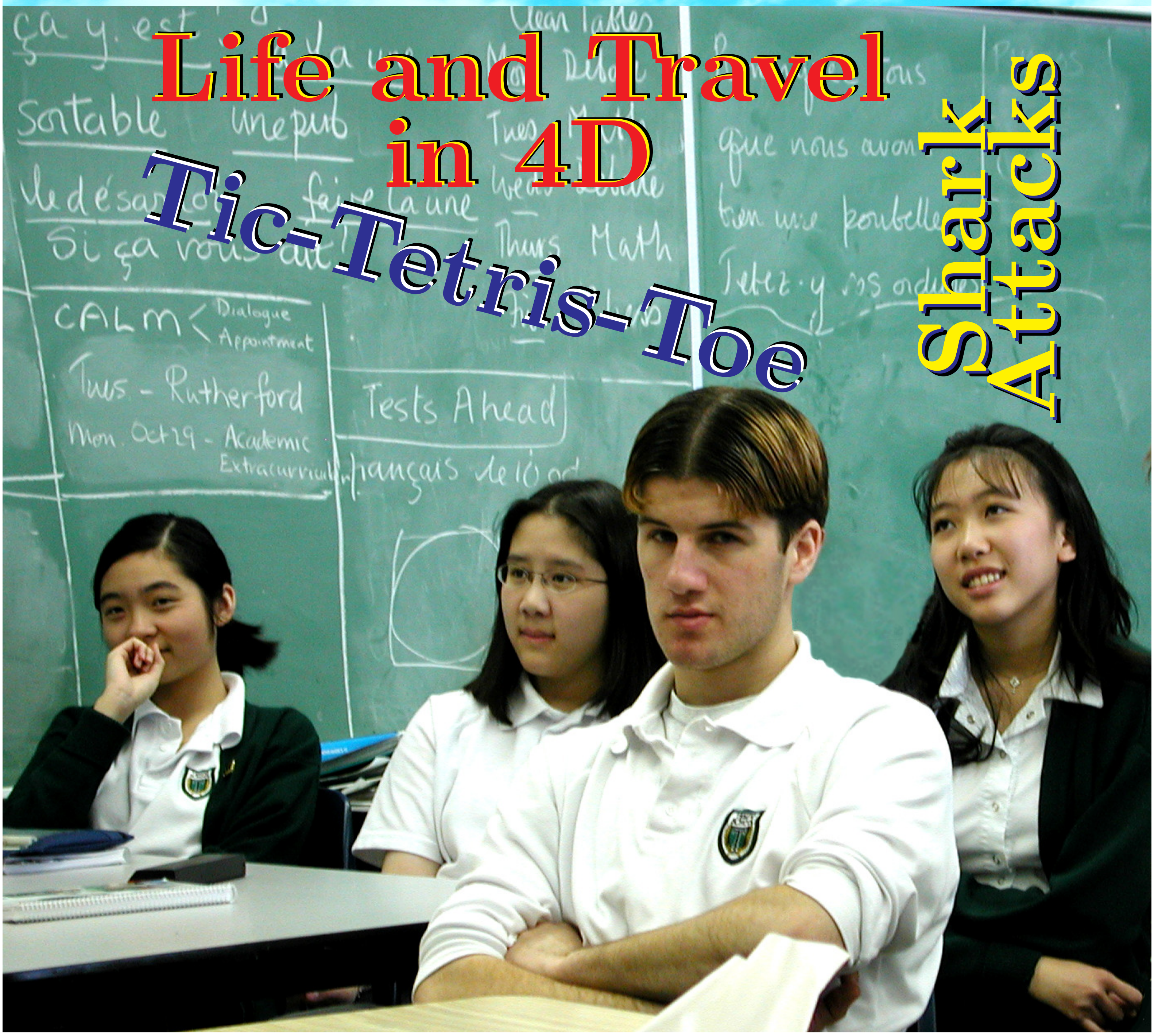
π in the Sky

The Pacific Institute for the Mathematical Sciences

December 2001



Life and Travel in 4D

Tic-Tetris-Toe

Shark Attacks

π in the Sky is a semi-annual publication of

The Pacific Institute for the Mathematical Sciences

PIMS is supported by the Natural Sciences and Engineering Research Council of Canada, the British Columbia Information, Science and Technology Agency, the Alberta Ministry of Innovation and Science, Simon Fraser University, the University of Alberta, the University of British Columbia, the University of Calgary, the University of Victoria, the University of Washington, the University of Northern British Columbia, and the University of Lethbridge.

This journal is devoted to cultivating mathematical reasoning and problem-solving skills and preparing students to face the challenges of the high-technology era.

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We also welcome **Letters to the Editor** from teachers, students, parents or anybody interested in math education (be sure to include your full name and phone number).

Cover Page: In October 2001, *π in the Sky* was invited by the principal of Edmonton's Tempo School, Dr. Kapoor, to meet with students in grades 10 and 11. The picture on the cover page was taken by Henry Van Roessel at Tempo School during our visit. More photos from that visit are published on page 28.

If you would like to see your school on the cover page of *π in the Sky*, please invite us for a short visit to meet your students and staff.

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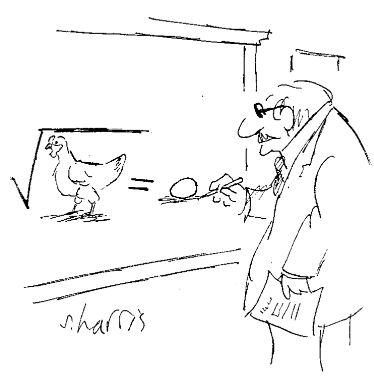

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Sidney Harris

Math Opinion!

This column is an open forum. We welcome opinions on all mathematical issues: research; education; and communication. Please feel free to write. Opinions expressed in this forum do not necessarily reflect those of the editorial board, PIMS, or its sponsors.

The Language of Mathematics

by Timothy Taylor

I'm a writer. I write stories—novels, short stories, novellas. I'm aware that sometimes the creative arts (like my fiction) and the hard sciences (like mathematics) are considered uncomfortable companions. People tend to imagine themselves as being attracted to one or the other, but not to both. In my case, to be truthful, I found mathematics difficult in school and for a long time I thought I didn't like the subject. But I was wrong. Not only have I surprised myself in recent years by discovering an interest in mathematics and its applications, I have surprised myself more by discovering this interest through my creative writing.

Somebody once asked me, "What's the most important skill required to write fiction?" I told them that you had to be able to sit alone in a room and type on a computer for long periods of time. This is partly true. When you have your story idea and you know your characters, there comes a point when you simply have to sit down and write for as long as it takes to finish. But that's not the only requirement, of course. There is also a lot of research required to prepare yourself. In my writing, some of this research might be considered *incidental*. If my character visits Rome, I make it my business to learn about the city. I don't just write, "He went to Rome." I write something like, "He stayed at the Albergo Pomezia in Campo di'Fiori not far from the old Jewish ghetto." This detail might not help us understand the character better, but it serves to create a sense of reality.

Ultimately, you do want the reader to understand the characters however, and here's where a more *integral* kind of research comes into play. Characters in my stories tend to have a fairly clear set of desires and objectives in life. These can range from grand to mundane. But in any case, there will be a particular set of issues that concern a character, some variety of problem that he or she must solve. For example, a character arriving at a new school might wish to meet new friends. My character in Rome (he's an art critic) is consumed with the work of a particular painter. The crucial thing is that the set of issues, or the problems that confront a character, determine which language they use.

When I say "language," I don't mean "tongue," where Chinese, Russian, English, or French might be examples. Instead, I mean the set of words and concepts that a character is inclined to use within a given tongue. These arise directly from the issues that concern that character. And so, there is a unique language of art criticism (colour, composition, theme, culture, aesthetic etc.), just as there is a language of business, of church, of personal relationships, and—drum roll please—of mathematics!

I hadn't really considered this until I wrote a story called *Silent Cruise* a few years back. In that story, I introduce Dett and Sheedy. Sheedy is a businessman and thinks only in those terms. Dett is a young man who is consumed by his own way of calculating probabilities (he does this, in part, because he likes betting at the racetrack). In order to put words into Dett's mouth that make sense given his very peculiar obsessions, I had to re-acquaint myself with a language I hadn't thought about in some time: mathematics. I should emphasize that Dett's way of making calculations is not rigourous. A math student reading the story would see this right away. But the point is that he thinks not in a literate language (like Sheedy), but in a numerate one. And Dett's way of expressing himself, to a large extent, defines who he is, how well he communicates with Sheedy, and what kinds of problems he is able or unable to solve. As I wrote the story, I enjoyed trying to "think" through Dett in his numerate way, even though I had a hard time doing so at school. In the process, I came to think that of all the languages I had researched for characters over the years, mathematics is very special. I would even go so far as to say that it is a *precious* language. It's difficult to learn and, as a result, it is rare and valuable. But it is also very powerful, and perhaps this interests me more. As I wrote Dett's story—even though he applied his numeracy in an unconventional way—he had the tools to solve many, many problems that Sheedy did not. And that fact, boiled right down, was the essence of my story.

An editor once commented on *Silent Cruise* in a newspaper article. He said that I had drawn a picture of a character who thought primarily with numbers and how that was very unusual in Canadian fiction. I take these words as a compliment beyond any other that I have received in connection with my writing. They mean that I had not only done my research well enough to convince this editor, but that I had communicated some of what is precious—rare and valuable—in the language of mathematics. I only wish I spoke it better.

Timothy Taylor is the author of the national bestseller *Stanley Park,* a novel. His book, *Silent Cruise and Other Stories,* will be published next year. The short story, *Silent Cruise,* was short-listed for the Journey Prize 2000. Taylor won the prize for a different story, where the language spoken was concerned mostly with cheese.

We should also mention that an article written by Timothy Taylor for Saturday Night Magazine won a Gold Medal at the National Magazine Awards.

Tic-Tetris-Toe

by Andy Liu

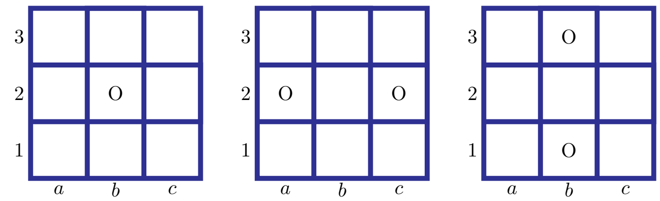


Figure 2

Part I: Introduction

Tic-Tetris-Toe is very much like *Tic-Tac-Toe*. The classic game is played on a 3 by 3 board, taking a square in each turn. Whoever is first to get 3 squares in a row or 3 on a diagonal wins. However, in this new game, we make two changes.

First, while we still play on a square board, it does not have to be 3 by 3. **Tic-Tetris-Toe** is actually five different games, each with a board of a different size. Second, we try to get different shapes. We use those from the popular video game *Tetris*, as shown in Figure 1.

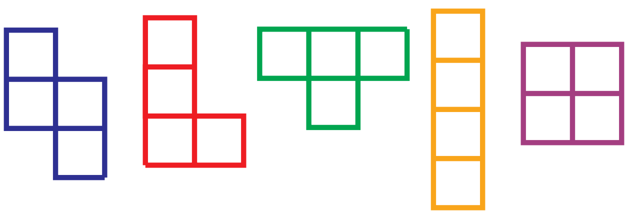


Figure 1

There are actually seven different pieces, but since they are allowed to turn over, we have only five **Tic-Tetris-Toe** games. We call these pieces N4, L4, T4, I4 and O4, because they each have four squares and look like the letters N, L, T, I and O, respectively.

For N4, we play on a 3 by 3 board. For L4, we play on a 4 by 4 board. For T4, we play on a 5 by 5 board. For I4, we play on a 7 by 7 board. For O4, we play on a 9 by 9 board. Of course, the advantage is with the first player. Can you figure out a way for a sure win if I let you go first? Try these games with your friends, and then check below. Don't peek—that will spoil the fun!

Part II: The N4 Game

Let us label the rows of the board 1, 2, and 3, and the columns a, b, and c. That way, each square will have a name. For example, the square at the bottom left corner will be called a1.

You will need at least four moves to win, and you will have an extra fifth move that may come in handy in some scenarios. Mapping out a winning strategy requires that you look quite far ahead. On the other hand, I (the opposing player) may be able to stop you from winning with one or two moves. Perhaps you should first consider what my strategy will be.

Figure 2 shows three ways in which I can stop you from winning. In each case, even if I give you all of the remaining squares, you still cannot complete N4. This tells you that you must take b2 in your first move, and make sure that you take at least one of b1 and b3, and at least one of a2 and c2.

Note that once you have taken b2, you do not have to worry about me sneaking up on you for a surprise win. You will do no worse than a draw. It would be too embarrassing to lose as the first player. Can I still stop you from winning? After you have taken b2, I really have two different choices: taking an edge square or a corner one.

Suppose I take a2. You already know that you must take c2. Now I give up. In your third move, you can take b1 and create a double-threat at a3 and c1. If I prevent you from doing this by taking one of these three squares, you will take b3 and create a double-threat at a1 and c3. Figure 3 shows that the key to your success is the W5 shape.

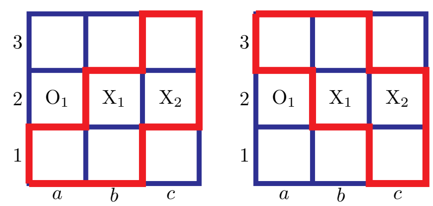


Figure 3

Am I better off if I start with a corner square, say a1? Suppose you still take c2. After all, it has worked once. Now I know that I must take one of a3, b3, and c1. You can force me to take c3 on my next move by taking b1 yourself. Then you can create a double-threat by taking one of c3, a2, and a3, depending on my move.

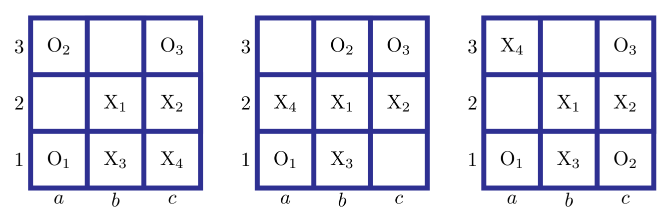


Figure 4

Can you remember all of this? You do not have to do that. Just understand that you must have b2, one of a2 and c2, and one of b1 and b3. Then look for double threats. With a little bit of practice, you will always win if you move first.

Part III: The L4, T4 and I4 Games

Label the extra rows 4, 5 and so on, and the extra columns d, e and so on. You can have an easy win in the L4 game. Start by taking b2. You are guaranteed to get b3 or c2 in your next move. If both are still there and both b1 and b4 are still empty, take b3. Otherwise, a2 and d2 will be empty, so take c2. On your third move, complete 3 squares in a row, and I cannot stop you from completing L4 on your fourth move. Since I have only made three moves so far, I cannot beat you to it.

You can win the T4 game by starting at the obvious place, c3. I can make one of five essentially different responses, at a1, a2, a3, b2, or b3. On your second move, you take d4. On your third move, you take either c4 to create a double-threat at b4 and c5, or d3 to create a double-threat at d2 and e3. I can neither stop you nor beat you to it.

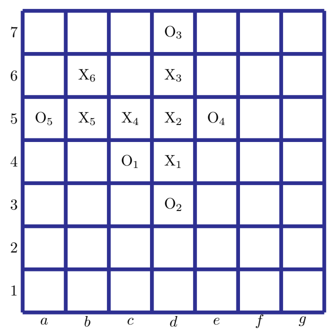


Figure 5

The I4 game is the only one of the five that can be played competitively. While you have a sure win, it cannot be forced until your eighth move. In trying for the win, it is possible that you may set up a double-threat for me. Figure 5 shows a sample game in which I put up a good fight. On the seventh move, you either take c6 for the double-threat at a6 and e6, or take b4 for the double-threat at b3 and b7. I cannot stop both.

Part IV: The O4 Game

This game holds a big surprise. Even though the board looks more than large enough, you will not be able to force a win. I have a very simple but effective counter strategy that will prevent you from winning, even if we play on an infinite board. It is an elegant idea which demonstrates the beauty of mathematics.

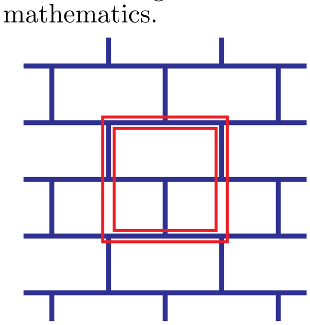


Figure 6

I will combine pairs of adjacent squares into dominoes in the pattern of a brick wall, into which you will bash your head in vain. Whenever you take a square, I will take the other square of the same domino. As shown in Figure 6, no matter how you fit in O4, it must contain a complete domino. Since you can only have half of it, you cannot win!

Part V: Further Projects

Problem 1.
Find four connected shapes of three squares or less, joined edge-to-edge.

Remark: These shapes are called the monomino O1, the domino I2, and the trominoes I3 and V3. None of them provides much challenge as a game—the first player has an easy win if the board is big enough. This is because each of these pieces form parts of other pieces for which the first player can win. Our *Tetris* pieces are the tetrominoes. If we go to the pentominoes, you will find these games much more challenging. There are twelve such pieces, called F5, I5, L5, N5, P5, T5, U5, V5, W5, X5, Y5, and Z5, as shown in Figure 7. *Pentomino* is a registered trademark of **Solomon Golomb**, who has written a wonderful book called *Polyominoes*. This word means, "shaped or formed of many squares." After the pentominoes come the 35 hexominoes, 108 heptominoes, 369 octominoes, and so on.

Problem 2.
Since P5 contains O4, and the domino strategy of Figure 6 works for the O4 game, the second player can also force a draw in the P5 game, even if it is played on an infinite board. On the other hand, there are four pentominoes that do not contain O4, but for which the domino strategy of Figure 6 also works. Which pentominoes are they?

Problem 3.
Show how the first player can force a win for the N5 game on a 6 by 6 board, and for the L5 and Y5 games on a 7 by 7 board.

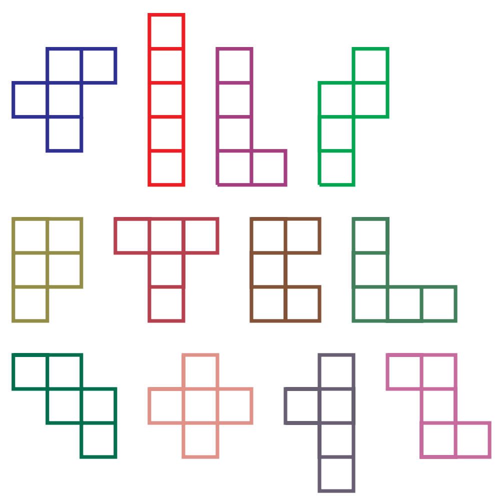


Figure 7

Problem 4.
Match each of the other four pentominoes with one of the patterns in Figure 8 for a domino strategy.

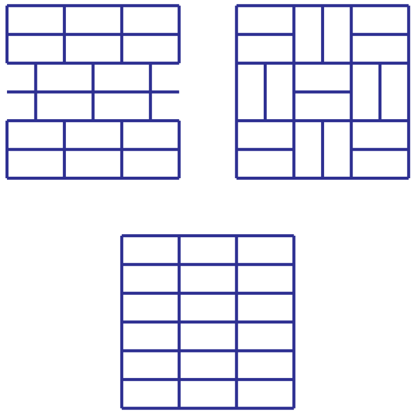


Figure 8

Problem 5.
How many of the hexominoes contain a pentomino for which the second player has a domino strategy?

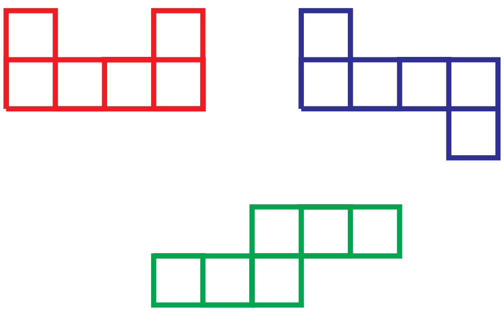


Figure 9

Problem 6.
Find domino strategies for the second player in games using the hexominoes in Figure 9. For one of them, you will have to find a new pattern.

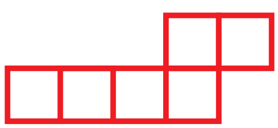


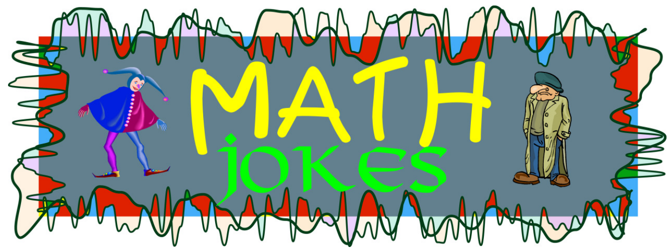
Figure 10

Problem 7.
With the possible exception of the hexomino in Figure 10, no polyominoes formed of six or more squares offer the first player a sure win, even on an infinite board. Can a win be forced in a game using this hexomino?

Problem 8.
Returning to *Tic-Tetris-Toe*, it is easy to see that there is no win for the first player in the N4 game if it is played on a 2 by 2 board, because it is not even big enough to hold the piece. The L3 game on a 3 by 3 board is also a draw, if played properly. The classic *Tic-Tack-Toe* is still a draw even with additional winning configurations. Can the first player still force a win in the T4 games on a 4 by 4 board, or the I4 game on a 6 by 6 board?

Part VI: Acknowledgement

This article is based on **Martin Gardner**'s Mathematical Games column in *Scientific American* magazine, April, 1979. It has since been collected into the anthology *Fractal Music, Hypercards and More*, as Chapter 13, under the title *Generalized Tic-tac-toe*. This book was published by W. H. Freeman and Company, New York, in 1992. The original work was done by the noted graph theorist **Frank Harary**.



A mother of three is pregnant with her fourth child. One evening, her eldest daughter says to her dad, "Do you know, daddy, what I've found out?"

"No."

"The new baby will be Chinese!"

"What?!"

"Yes. I've read in the paper that statistics show that every fourth child born nowadays is Chinese. . . ."

A father who is very much concerned about his son's poor grades in math decides to register him at a religious school. After his first term there, the son brings home his report card; he gets 'A's in math.

The father is, of course, pleased, but wants to know, "Why are your math grades suddenly so good?"

"You know," the son explains, "when I walked into the classroom the first day and saw that guy nailed to a plus sign on the wall, I knew one thing—this place means business!"

"What happened to your girlfriend, that really smart math student?"

"She is no longer my girlfriend. I caught her cheating on me."

"I don't believe that she cheated on you!"

"Well, a couple of nights ago I called her on the phone, and she told me that she was in bed wrestling with three unknowns. . . ."

Q: Why do mathematicians often confuse Christmas and Halloween?

A: Because Oct 31 = Dec 25.*

Q: How do you make one burn?

A: Differentiate a log fire.

* Write to us if you get this joke!

Weierstraß*

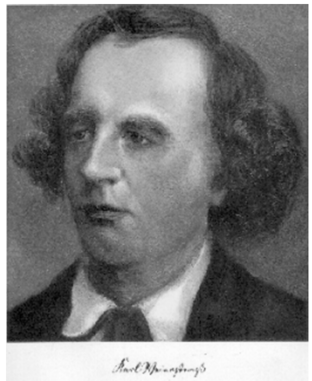
Volker Runde†

Each university and each department develops a peculiar kind of folklore—anecdotes about those of its graduates (or dropouts) that somehow managed to become famous (or notorious). Very often, there is an element of glee to these stories: "Well, he may now be a government minister, but *I* flunked him in calculus!" And also very often, it is impossible to tell the truth from the legend.

Karl Weierstraß

When I was a math student at Münster, Germany in the 1980s, such anecdotes centered mainly around two people: Gerd Faltings, the first and only German to win the Fields medal[1], mathematics' equivalent of the Nobel prize; and Karl Weierstraß, the man who (besides many other mathematical accomplishments) introduced ε and δ into calculus. Weierstraß had been a student at Münster in the 1830s and 1840s. There was no one around anymore who knew anybody who had known anybody who had known anybody who had known Weierstraß, but this didn't prevent the folklore from blooming. According to his legend, Weierstraß flunked out of law school because he spent most of his time there drinking beer and doing mathematics. Then he worked for more than ten years as a school teacher in remote parts of Prussia, teaching not only mathematics, but also subjects like botany, calligraphy, and physical education. Finally, when almost 40 years old, he became a famous mathematician, and was eventually appointed a professor at Berlin—without ever having received a PhD. This story may sound wild, and in some ways it simplifies the facts, but it is not far from the truth.

Karl Theodor Wilhelm Weierstraß was born on October 31, 1815, in the village of Ostenfelde, which is located in what was then the Prussian province of Westphalia. A street and an elementary school in Ostenfelde are named after him, and his birth house—still occupied today—is listed in a local tourist guide. His father, who worked for Prussia's customs and taxation authorities, was sent from one post to the next within short periods of time. For the first 14 years of Karl Weierstraß' life, his family was more or less constantly on the move. In 1829, Karl's father obtained an assistant's position at the tax office in the city of Paderborn (also in Westphalia), and the family could finally settle down. Young Karl enrolled at the local Catholic *Gymnasium* in Paderborn, where he excelled not only in mathematics, but also in German, Latin, and Greek. Not only was he a strong student, he was also quite capable of putting his brains to work on much more practical matters. At age 15, Karl contributed to his family's income by doing bookkeeping for a wealthy merchant's widow.



Karl Weierstraß as a young man

Throughout his life, Weierstraß Senior suffered from the knowledge that he did not have the right education to rise to a rank in the Prussian civil service that would have better suited his abilities. Instead, he had to content himself with relatively low-level positions, not very challenging and not very well paid. Like many a father in this situation, he was determined to prevent such a fate befalling his bright eldest son. When Karl graduated in 1834, his father decided to send him to Bonn to study *Kameralistik* (a combination of law, finance, and administration). Being a dutiful son, Karl went...

...and did all he could to sabotage the life his father had planned for him. He joined a *schlagende Verbindung*, a kind of student fraternity typical of German universities in the 19th century. Besides keeping the brewing industry busy, fraternity members engaged in a peculiar ritual: the *Mensur*, a swordfight with a peculiar twist. Unlike in today's athletic competitions, the students fought with sharp sabers. They wore protective gear that covered most of their bodies—except the cheeks. During a *Mensur*, the opponents tried to inflict gashes on one another's cheeks. The scars were borne with pride as signs of honour and manhood.[2] Almost two metres tall, quick on his feet, and with strong arms, Karl Weierstraß was a fearsome swordsman. His face remained unscarred, and after a while nobody was keen on challenging him anymore. Having escaped from under his father's tutelage, he spent his years at Bonn drinking beer and wielding the saber—and seriously studying mathematics. Although he was not enrolled in mathematics, he read some of the most advanced math books of his time. In 1838, when it was time for him to take his exams, he simply dropped out.

His family was desperate. They had made considerable financial sacrifices to secure a better future for Karl, who had let them down. Having wasted four years of his life, he needed a bread-winning degree, and fast. So, in 1839,

* The last letter is not a β, but an "ß", a letter unique to the German alphabet, which is pronounced like an "s". Books written in English usually spell the name "Weierstrass."

† Volker Runde is a professor in the Department of Mathematical Sciences at the University of Alberta. His web site is http://www.math.ualberta.ca/~runde/runde.html and his E-mail address is vrunde@ualberta.ca.

[1] If you want to know more about the Fields medal: there is an article on it—*The Top Mathematics Award* by Florin Diacu—in the June 2001 issue of π *in the Sky*.

[2] If you find such ideas of honour and manhood absolutely revolting, you're absolutely right.

he enrolled at the *Akademie* in Münster, the forerunner to today's university, to become a secondary school teacher. Although this was not really a university, but rather a teacher training college, they had one good mathematician teaching there—Christoph Gudermann. He is said to have been an abysmal teacher: very often, he had just one student sitting in his class—Karl Weierstraß. In 1840, Weierstraß graduated. His thesis was so good that Gudermann believed it to be strong enough for a doctoral degree. However, the *Akademie* was not *really* a university; it did not have the right to grant doctorates. So, instead of receiving a doctorate and starting an academic career, Weierstraß left the *Akademie* as a mere school teacher.

His first job (probationary) was in Münster. One year later, he was sent to Deutsch Krona[3] in the province of West Prussia as an auxiliary teacher, then, in 1848, to Braunsberg[4] in East Prussia. Of course, he taught mathematics, but also physics, geography, history, German, and—believe it or not—calligraphy and physical education. Besides the demands of working full time as a teacher and having a social life (remember, he liked beer), he found time to do research in mathematics. During his time in Braunsberg, he published a few papers in his school's yearbook. High school year books are not exactly where people look for cutting edge research in mathematics, and so nobody noticed them. Then, in 1854, he published a paper entitled, "Zur Theorie der Abelschen Functionen" in a widely respected journal. I won't even make an attempt to explain what it was about. But unlike his previous work, this one *was* noticed.

It dawned on mathematicians all over Europe that the man who was probably the leading analyst of his day was rotting in a small East Prussian town, spending most of his time teaching youngsters calligraphy and physical education. On March 31, 1854, Weierstraß finally received a doctorate, an honorary one from the university of Königsberg.[5] In 1856, he accepted a position at the *Gewerbeinstitut* in Berlin, an engineering school, and a year later he joined the faculty of the University of Berlin as an adjunct professor. As a teacher, he attracted large audiences. Often, he taught in front of more than 200 students. In 1869, when he was almost 50 years old, Weierstraß was appointed full professor at the university of Berlin. In 1873 and 1874, he was *Rektor magnificus* of the university; in 1875, he became a knight of the order "Pour le Mérite" in the category of Arts and Sciences, the highest honour newly unified Germany could bestow upon one of its citizens; and, in 1885, on the occasion of his 70th birthday, a commemorative coin was issued in his honour.

The years of leading a double life as a secondary school teacher and a mathematical researcher took their toll on Weierstraß' health. A less vigorous man would probably have collapsed under the double burden much earlier. In 1850, Weierstraß began to suffer from attacks of dizziness, which culminated in a collapse in 1861. He had to pause for a year before he could teach again, and he never recovered fully. In 1890, at age 75, Weierstraß retired from teaching because of his failing health. The last years of his life were spent in a wheelchair. In 1897, he died.

Weierstraß published few papers—he was very critical toward his own work. But although he was a brilliant researcher, the greatest impact he had on mathematics was as a teacher. At Berlin, he repeatedly taught a two-year course on analysis, the predecessor of all modern introductions to calculus and analysis. Although he never wrote a textbook, notes taken in class by his students have survived and convey an impression of his lectures. Perhaps the longest lasting legacy of those lectures is their emphasis on rigour. When calculus was created in the 17th century, mathematicians did not worry about rigourously proving their results. For example, the first derivative dy/dx of a function $y = f(x)$ was thought of as a quotient of two "infinitesimals" (i.e., infinitely small quantities dy and dx). Nobody could really tell what infinitely small quantities were supposed to be, but mathematicians then didn't really care. The new mathematics enabled them to solve problems in physics and engineering that had been beyond the reach of the human mind before. So why bother with rigour? In the 18th century, mathematicians went so far as to proclaim that rigour was for philosophers and theologians, not for mathematicians. But with the lack of rigour, contradictory results cropped up with disturbing frequency—people often arrived at formulae that were obviously wrong. And, if a particular formula determines whether or not a bridge collapses, you don't want it to be wrong. Weierstraß realized that if calculus was to rest on solid foundations, its central notion, that of the limit, had to be made rigourous. He introduced the definition that (essentially) is still used today in classrooms:

> *A number y_0 is the limit of a function $f(x)$ as x tends to x_0 if, for each $\epsilon > 0$, there is $\delta > 0$ such that $|f(x) - y_0| < \epsilon$ for each x with $|x - x_0| < \delta$.*

Students may curse it, but it will not go away.

Weierstraß was not only an influential lecturer, but also one of the most prolific advisors of PhD theses of all time. There is a database on the Internet[6] that lists 31 PhD students of Weierstraß and 1,346 descendants (i.e., PhDs of PhDs of PhDs etc.) of Weierstraß. Interestingly, the two former students who generated the most folklore weren't his students in a technical sense.

Sofya Kovalevskaya was a young Russian noblewoman who had come to Germany to study mathematics. This alone was no small feat at a time when the very idea of a woman receiving a university education was revolutionary. For two years, she studied at Heidelberg, where authorities would not let her enroll officially, but eventually allowed her to attend lectures unofficially (provided the instructor did not object). Then she moved to Berlin to work with Weierstraß, only to find that she was not even allowed to audit lectures.

[3] Now Wałcz in Poland.
[4] Now Braniewo in Poland.
[5] Now Kaliningrad in Russia.

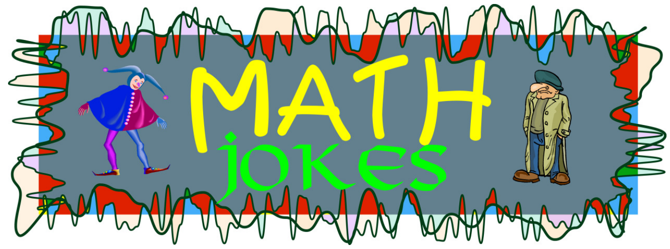
[6] The Mathematics Genealogy Project at http://hcoonce.math.mankato.msus.edu/ .

Sofya Kovalevskaya

This prompted Weierstraß, by all we know a politically conservative man, to tutor her privately. Since Kovalevskaya could not receive a doctorate from Berlin, Weierstraß used his influence to persuade the University of Göttingen to award her the degree in 1874. She spent the following nine years jobhunting. Being a woman didn't help. The best job she could find was teaching arithmetic at an elementary school. Finally, in 1883, she was offered a professorship at Stockholm, where she worked until her death in 1891 at the age of 41. Weierstraß and Kovalevskaya stayed in touch throughout her mathematical career. After her death, Weierstraß destroyed their correspondence. This fact, along with Kovalevskaya's striking beauty, gave rise to innuendos that she may have been more to Weierstraß (who never married) than just a student. Perhaps—but we don't know.

Karl Weierstraß in old age

Gösta Mittag-Leffler, another of the great mathematician's protegées, was also not Weierstraß' student strictly speaking. Already enrolled at the University of Uppsala, Sweden, he came to Berlin in 1875 to attend Weierstraß's lectures, which had an enormous impact on his mathematical development. He then returned to his native Sweden, where he received his doctorate. Over the years, Mittag-Leffler became indisputably the most influential mathematician of his time in Sweden. He made use of his clout to overcome the obstacles faced by Sofya Kovalevskaya regarding her appointment at Stockholm. What Mittag-Leffler is most famous for, however, is not a mathematical accomplishment, but a piece of mathematical folklore. To this day, mathematicians suffer quietly from the lack of a Nobel prize, and, some say, Mittag-Leffler is to blame—according to legend, the first version of Nobel's will mentioned a prize in mathematics. Then, Nobel found out that his wife had had an affair with Mittag-Leffler. Infuriated that his wife's lover could well be the first prize winner, Nobel changed his will and removed the math prize. That's a fine piece of juicy folklore, but nothing more; like Weierstraß, Nobel was a lifelong bachelor.



At the end of his course on mathematical methods in optimization, the professor sternly looked at his students and said, "There is one final piece of advice I'm going to give you now—whatever you have learned in my course, never, ever try to apply it to your personal lives!"

"Why?" the students asked.

"Well, some years ago, I observed my wife preparing breakfast, and I noticed that she wasted a lot of time walking back and forth in the kitchen. So, I went to work, optimized the whole procedure, and told my wife about it."

"And what happened?"

"Before I applied my expert knowledge, my wife needed about half an hour to prepare breakfast for the two of us. And now, it takes *me* less than fifteen minutes. . . ."

Q: What is an extroverted mathematician?

A: One who, in conversation, looks at the other person's shoes instead of at his own.

In a dark, narrow alley, a function and a differential operator meet, "Get out of my way or I'll differentiate you 'til you're zero!"

"Try it—I'm e^x. . . ."

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Sidney Harris

Same alley, same function, but a different operator: "Get out of my way or I'll differentiate you 'til you're zero!"

"Try it—I'm e^x. . . ."

"Too bad. . . I'm d/dy."

Life and Travel in 4D

by Tomasz Kaczynski[†]

> *It is the spirit of the age to believe that any fact, no matter how suspect, is superior to any imaginative exercise, no matter how true.*
> Gore Vidal

In higher-level mathematics courses, the study of vector spaces of arbitrary dimension n or even infinite dimension is often required. A natural question that arises is:

"How can we visualize a space of a dimension higher than 3?"

We can draw pictures in 2D, we can make models in 3D—but 4D? That seems very strange! Most teachers have no choice but to introduce an n-dimensional space $\mathbb{R}^n$ as a purely algebraic object consisting of sequences

$$\vec{x} := (x_1, x_2, x_3, \ldots, x_n)$$

of n real numbers. But this does not satisfy people who dabble more in geometry than in algebra. Some physicists and mathematicians claim to be able to see in 4D—Einstein supposedly could even see in 5D! I do not quite believe those stories, but rather think that all we can do is accept 4D or 5D, learn to cope with it, and maybe one day we will believe that we really see it. When I was in college, I once attended a series of talks given by university students with the aim of popularizing mathematics. There I learned a beautiful way of coping with a multi-dimensional space. The answer is in the concept of empathy, the capacity for participation in another's feelings or ideas. Who is that other person? He is called *Flatman*.[*]

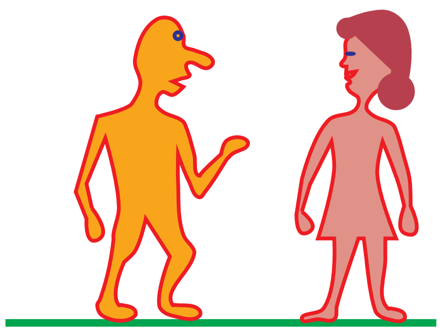
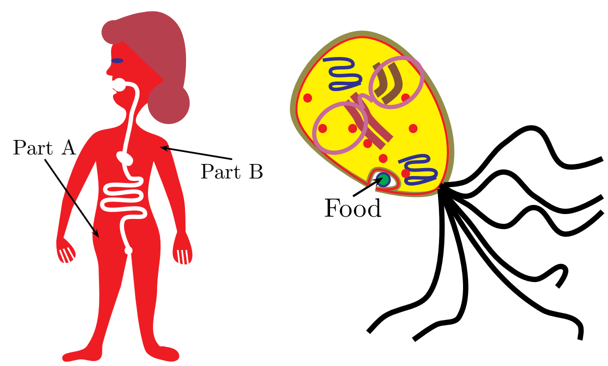


Figure 1

Let us put ourselves in the position of *Flatman*, a 2D man who lives in a 2D world and is unable to imagine a 3D space. What does his 2D world look like? His world is a straight line. His skin is not a surface, it is a closed curve. This is shown in Figure 1, which is not just another 2D projection of our World, but an image showing how his Flat World would actually appear.This presents Flatman with some technical problems. For example, if Flatman wants to pass to the right of Flatwoman, he has to jump over her. The anatomy of Flatman or Flatwoman is also not evident.

Now, let's look at Figure 2. Flatwoman's digestive system cannot be a tube like the one we have inside us, because it would split her into at least two separate parts (called by mathematicians *connected components*), as shown on Figure 2(a). We may guess that Flatman or Flatwoman must digest food similarly to the bacteria in Figure 2(b).

Note that the Flatman in Figure 1 cannot see the *interior* of the Flatwoman. But we, 3D-beings, can see her interior from our outer dimension. Maybe some superior 4D-being can see what we have inside without using X-rays, and could remove a tumor from a patient's body without any surgery. It is thus clear that concepts such as *interior*, *exterior*, and *boundary* are relative to the outer space in which our world is embedded.



(a) (b)

Figure 2

Let's give this discussion some mathematical meaning. Suppose a prisoner is kept at a point $P = (0, 0)$ of a 2D-world $\mathbb{R}^2$ whose points are denoted by the Cartesian coordinates (x, y). His cell is limited by the circle S^1 given by the equation $x^2 + y^2 = 1$. If there is no hole in the circle, there is no way he could get outside of the circle, e.g., to the point $Q = (2, 0)$. This fact is intuitively acceptable, but very hard to prove—it is the famous *Jordan Closed Curve Theorem*. But everything changes if we embed the plane into the 3D-space $\mathbb{R}^3$, whose points are denoted by Cartesian coordinates (x, y, z). Our planar world is given by the equation $z = 0$. The prisoner's position is now $P = (0, 0, 0)$, the boundary circle is given by the pair of equations

$$\begin{aligned} x^2 + y^2 &= 1, \\ z &= 0, \end{aligned}$$

and the destination point is $Q = (2, 0, 0)$. The prisoner can easily get out by jumping over the circle. A possible

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[*] We recommend the book by Edwin A. Abbott, "Flatland: a romance of many dimensions," London, Seeley, 1884.

trajectory in time t can be given by

$$\begin{aligned} x &= t, \\ y &= 0, \\ z &= t(2-t), \end{aligned}$$

which starts at $t = 0$ and reaches the point Q at $t = 2$. This is shown in Figure 3(a).

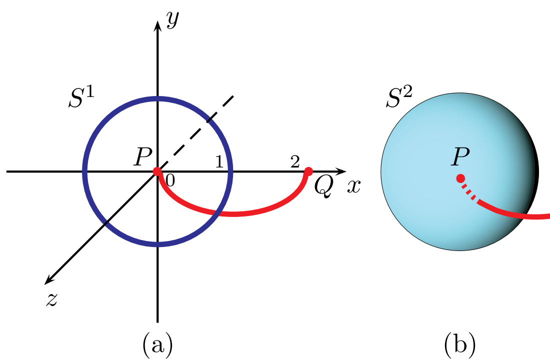


Figure 3

We may now raise the dimension. Consider the point $P = (0, 0, 0)$ inside a balloon-shaped cell called *sphere* S^2 and given by the equation $x^2 + y^2 + z^2 = 1$. Consider the point $Q = (2, 0, 0)$ on the outside of the sphere. Again, there is no way a prisoner staying at P could move out to Q without cutting through the sphere. Let's add an extra dimension: the points of $\mathbb{R}^4$ are (x, y, z, u), $P = (0, 0, 0, 0)$, and $Q = (2, 0, 0, 0)$. Our 3D-space is given by the equation $u = 0$, and the limiting 2D-sphere S^2 is now given by

$$\begin{aligned} x^2 + y^2 + z^2 &= 1, \\ u &= 0. \end{aligned}$$

With the extra fourth dimension, the prisoner can jump outside the sphere without cutting it. The trajectory is now

$$\begin{aligned} x &= t, \\ y &= 0, \\ z &= 0, \\ u &= t(2-t). \end{aligned}$$

This scenario is shown in Figure 3(b). Now, let's tackle a more serious topic. Once upon a time, a scientist named Flatilei discovered that the Earth is not a line, but a circle around a disc. Many flat sailors rushed to attempt a cruise around the world, the first one being Flatellan. Several centuries pass before some flastronomers claim that the universe is not a 2D space; it is actually limited. It might be, for example, a huge balloon in a 3D space, i.e., the sphere S_R^2 given by the equation

$$x^2 + y^2 + z^2 = R^2,$$

where R is the radius of the sphere. This is shown in Figure 4.

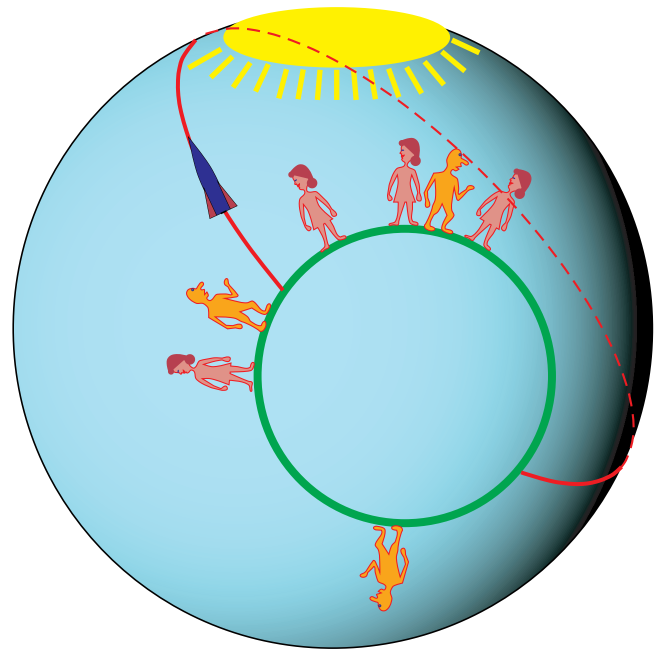
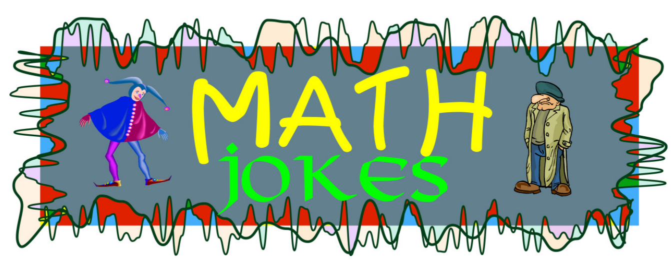


Figure 4

The flat circular Earth is a small round patch in that spherical universe. Science-fiction writers imagine stories of space missions where a spacecraft is sent along a straight line and returns to the Earth. How can that be? Because what was believed to be a straight line is actually a great circle on the sphere. We may attempt the same mental exercise by adding one more dimension, and view our universe as a 3D-sphere S^3 given by the equation

$$x^2 + y^2 + z^2 + t^2 = R^2.$$

One may ask: If the universe is limited, why must it be a 3D-sphere? Actually, it does not have to be. In the next issue, we will investigate other possibilities; look forward to the article *Travelling on the Surface of a Giant Donut.*



Q: Do you already know the latest stats joke?
A: Probably. . . .

Q: What is the fundamental principle of engineering mathematics?
A: Every function has a Taylor series that converges to the function and breaks off after the linear term.

Shark Attacks and the Poisson Approximation

by Byron Schmuland[†]

A story with the dramatic title *"Shark attacks attributed to random Poisson burst"* appeared on September 7, 2001 in the National Post. In the story, Professor David Kelton of Penn State University used a statistical model to try to explain the surprising number of shark attacks that occurred in Florida last summer. According to the article, Kelton suggests the spate of attacks may have had nothing to do with changing currents, dwindling food supplies, the recent rise in shark-feeding tourist operations, or any other external cause.

"Just because you see events happening in a rash like this does not imply that there's some physical driver causing them to happen. It is characteristic of random processes that they exhibit this bursty behaviour," he said.

What was the professor trying to say? Can mathematics really explain the increase in shark attacks? And what are the mysterious Poisson bursts?

The main point of the Professor Kelton's comments was that unpredictable events, like shark attacks, do not occur at regular intervals as in Figure 1(a), but tend to occur in clusters, as in Figure 1(b). The unpredictable nature of these events means that there are bound to be periods with a higher than average number of events, as well as periods with a lower than average number of events, or even no events at all.

[†] Find more about the author at the following web site:
http://www.stat.ualberta.ca/people/schmu/dept_page.html
You can also send your comments directly to the author at
schmu@stat.ualberta.ca

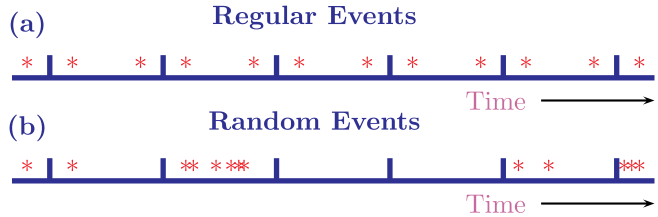


Figure 1

The statistical model used to study the sequences of random events gets its name from French mathematician Siméon Denis Poisson (1781-1840), who first wrote about the *Poisson distribution* in a book on law. The Poisson distribution can be used to calculate the chance that a particular time period will exhibit an abnormally large number of events (Poisson burst), or that it will exhibit no events at all. Since Poisson's time, this distribution has been applied to many different kinds of problems, such as the decay of radioactive particles, ecological studies on wildlife populations, traffic flow on the Internet, etc. Here is the marvellous formula that helps to predict the probability of random events:

The chance of exactly k events occurring is

$$\frac{\lambda^k}{k!} \times e^{-\lambda},$$

for $k = 0, 1, 2, \ldots$

The funny looking symbol λ is the Greek letter lambda, and it means "the average number of events." The symbol $k! = k \times (k-1) \times \ldots \times 2 \times 1$ is the factorial of k, and $e^{-\lambda}$ is the exponential function e^x with the value $x = -\lambda$ plugged in. Let's take this new formula out for a spin.

Shark Attack!

If, for example, we average two shark attacks per summer, then the chance of having six shark attacks next summer is obtained by plugging $\lambda = 2$ and $k = 6$ into the formula above. This gives

$$\text{probability of six attacks} \approx (2^6/6!) \times e^{-2} = 0.01203,$$

which is a little more than a 1% chance. This means that six shark attacks are quite unlikely in any one year, although it is likely to happen about once every 85 years. The chance that the whole summer passes without any shark attacks can also be calculated by plugging $\lambda = 2$ and $k = 0$ into the formula. This gives

$$\text{probability of no attacks} \approx (2^0/0!) \times e^{-2} = 0.13533,$$

which is a 13% chance. Thus, we can expect a "sharkless summer" every seven or eight years.

In this hypothetical shark problem, the number of attacks followed the Poisson distribution exactly. The Poisson distribution is most often used to find approximate probabilities in problems with n repeated trials and probability p of success. Let me show you what I mean.

Lotto 6–49

One of my favourite games to study is Lotto 6–49. Six numbers are randomly chosen from 1 to 49, and if you match all six numbers you win the jackpot. Since the number of possible ticket combinations is $\binom{49}{6} = 13{,}983{,}816$, your chance of winning the jackpot with one ticket is one in 13,983,816, which is $p = 7.15 \times 10^{-8}$. Let's say you are a regular Lotto 6–49 player and that you buy one ticket twice a week for 100 years. The total number of tickets you buy is $n = 100 \times 52 \times 2 = 10{,}400$. What is the chance that you will win a jackpot sometime during this 100 year run?

This is a pretty complex problem, but the Poisson formula makes it simple. First of all, the **average** number of jackpots during this time period is $\lambda = np = 10{,}400/13{,}983{,}816 = 0.0007437$. Plugging this into the formula with $k = 0$ shows that the chance of a "jackpot-less 100 years" is

$$\text{probability of no jackpots} \approx e^{-0.0007437} = 0.99926.$$

Wow! Even if you play Lotto 6–49 religiously for 100 years, there is a better than 99.9% chance that you will never, ever win the jackpot.

Coincidences

Take two decks of cards and shuffle both of them thoroughly. Give one deck to a friend and place both decks face down. Now, at the same time, you and your friend turn over the top cards. Are they the same? No? Then try again with the second card, the third card, etc. If you go through the whole deck, what is the chance that, at some point, you and your friend will turn over the same card?

In this problem, there are $n = 52$ trials and the chance of a success (coincidence) on each trial is $p = 1/52$. The average number of coincidences is $\lambda = np = 52/52 = 1$, and so putting $k = 0$ in the Poisson formula gives

$$\text{probability of no coincidences} \approx e^{-1} = 0.36788.$$

The chance that you **will** see a coincidence is $1 - 0.36788 = 0.63212$. You will get a coincidence about 63% of the time you play this game. Try it and see!

Birthday Problem

Suppose there are N people in your class. What are the odds that at least two people share a birthday? Imagine moving around the class checking every pair of people to see if they share a birthday. The number of trials is equal to the number of pairs of people, i.e., $n = \binom{N}{2} = N(N-1)/2$. The probability of success in a given trial is the chance that two randomly chosen people share a birthday, i.e., $p = 1/365$. This gives the average number of shared birthdays as $\lambda = N(N-1)/(2 \times 365)$, so the probability of "no shared birthdays" is

$$\text{probability of no shared birthdays} \approx e^{-N(N-1)/(2\times 365)}.$$

Therefore, the probability of at least one shared birthday is approximately $1 - e^{-N(N-1)/(2\times 365)}$. Here's what happens when you try using different values of N in this formula.

Probability of a shared birthday

N	Prob	N	Prob
10	0.115991	60	0.992166
20	0.405805	70	0.998662
30	0.696320	80	0.999826
40	0.881990	90	0.999983
50	0.965131	100	0.999999

With $N = 10$ people, there is only about an 11.5% chance of a shared birthday, but with $N = 30$ people there is a 69.6% chance. In a large class (like at a university!) with $N = 100$ students, a shared birthday is 99.9999% certain.

In a large class, perhaps it is possible to have a triple birthday. Following the same pattern, let's work out the chance that there is at least one triple shared birthday in a class of N people. This time, as each triple of people is checked, there are $\binom{N}{3} = N(N-1)(N-2)/6$ trials, and the chance of success on each trial is $p = 1/365^2$.

This gives $\lambda = N(N-1)(N-2)/(6 \times 365^2)$, so the probability of "no triple shared birthdays" is

$$\left.\begin{array}{c}\text{probability of no}\\ \text{triple shared birthdays}\end{array}\right\} \approx e^{-N(N-1)(N-2)/(6\times 365^2)}.$$

Therefore, the probability of at least one triple shared birthday is $1 - e^{-N(N-1)(N-2)/(6\times 365^2)}$. Now, let's look at different values of N in this formula.

Probability of a triple shared birthday

N	Prob	N	Prob
10	0.002699	60	0.537254
20	0.025344	70	0.708481
30	0.087370	80	0.842779
40	0.199470	90	0.929027
50	0.356838	100	0.973779

My large first year statistics courses usually have about 100 students, and I always check their birthdays. According to the table, there should be a triple shared birthday more than 97% of the time. It really is true; there has always been a triple shared birthday in my classes.

The Great One

During Wayne Gretzky's days as an Edmonton Oiler, he scored a remarkable 1669 points in 696 games, for a rate of $\lambda = 1669/696 = 2.39$ points per game. Using the Poisson formula, with $k = 0$, we estimate that the probability of Gretzky having a "pointless game" is

$$\text{probability of no points} \approx \frac{(2.39)^0}{0!}\, e^{-2.39} = 0.0909.$$

Over 696 games, this ought to translate into about $696 \times 0.0909 = 63.27$ pointless games. In fact, during that period, he had exactly 69 pointless games.

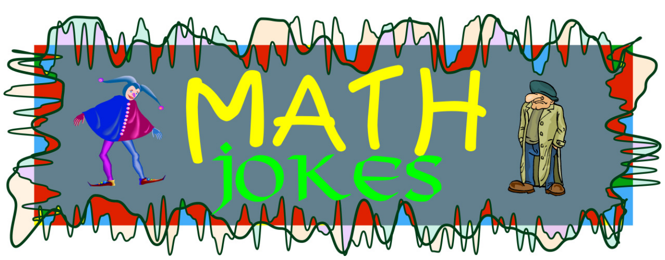
For one-point games, we find an approximate probability of

$$\text{probability of one point} \approx \frac{(2.39)^1}{1!} e^{-2.39} = 0.2180,$$

for a predicted value of $696 \times 0.2180 = 151.71$ one-point games. Let's try the same calculation for other values of k, and compare the Poisson formula prediction to the actual statistics.

Points	*Actual # Games*	*# Predicted by Poisson*
0	69	63.27
1	155	151.71
2	171	181.90
3	143	145.40
4	79	87.17
5	57	41.81
6	14	16.71
7	6	5.72
8	2	1.72
9	0	0.46

As you can see, there is remarkable agreement between the predictions of the Poisson formula, and the actual number of games with different point totals. This shows that Gretzky was not only a high scoring player, but a consistent one as well. The occasional pointless game, or "Poisson burst" in seven- or eight-point games, was not due to inconsistent play, but was exactly what would be expected in any random sequence of events. Another reason why he really was the Great One!



A physicist, a statistician, and a pure mathematician go to the races and place bets on horses.

The physicist's horse comes in last. "I don't understand it. I have determined each horse's strength through a series of careful measurements."

The statistician's horse does a little bit better, but still fails miserably. "How is this possible? I have statistically evaluated the results of all races for the past month."

They both look at the mathematician, whose horse came in first. "How did you do it?"

"Well," he explains. "First, I assumed that all horses were identical and spherical. . . ."

Two men are having a good time in a bar. Outside, there's a terrible thunderstorm. Finally, one of the men thinks that it's time to leave. Since he has been drinking, he decides to walk home.

"But aren't you afraid of being struck by lightning?" his friend asks.

"Not at all. Statistics shows that, in this part of the country, one person per year gets struck by lightning—and that one person died in the hospital three weeks ago."

"Isn't statistics wonderful?"

"How so?"

"Well, according to statistics, there are 42 million alligator eggs laid every year. Of those, only about half get hatched. Of those that hatch, three-fourths of them get eaten by predators in the first 36 days. And of the rest, only 5 percent get to be a year old for one reason or another. Isn't statistics wonderful?"

"What's so wonderful about all that?"

"If it weren't for statistics, we'd be up to our arses in alligators!"

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Do you know that 87.166253% of all statistics claim a precision that is not justified by the method employed?

A mathematician has been invited to speak at a conference. His talk is announced as, *Proof of the Riemann Hypothesis*.

When the conference takes place, he speaks about something completely different. After his talk, a colleague asks him, "Did you find an error in your proof?"

He replies: "No, I never had one."

"But why did you make this announcement?"

"That's my standard precaution—in case I die on my way to the conference. . . ."

The Rose and the Nautilus

A Geometric Mystery Story

by Klaus Hoechsmann[†]

The image below—or some variation thereof—appears in many books and web sites. The story behind it deals with such diverse concepts as

golden rectangles,
Fibonacci numbers,
regular pentagons,
and logarithmic spirals.

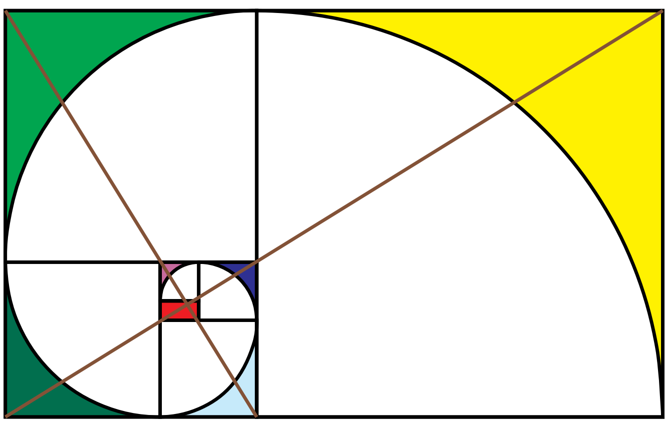


Figure 1

Its main appeal lies in uncovering the invisible threads connecting these items. Its origins are lost in the mists of antiquity, with some parts as old as the Pyramids, and others surfacing in Euclid's Elements (250 BC) and Ptolemy's *Almagest* (150 AD). The artwork by Leonardo da Vinci in *Pacioli's Divina Proportione* (1509 AD) helped spread its fame beyond the mathematical crowd.

Older accounts are entirely geometric, but with their fussy monochrome diagrams and awkward notation, they are difficult to follow. More recent versions, on the other hand, tend to achieve brevity by abundant use of algebraic formulas, which most people find incomprehensible. This article will try to provide a simple but complete account in the geometric vein and in full colour.

Figure 2 is a simplification of Figure 1. To recreate the latter, you need only cut a quarter circle out of the yellow square, another one out of the light green square, then the dark green square, and so on—and draw the brown diagonals in again.

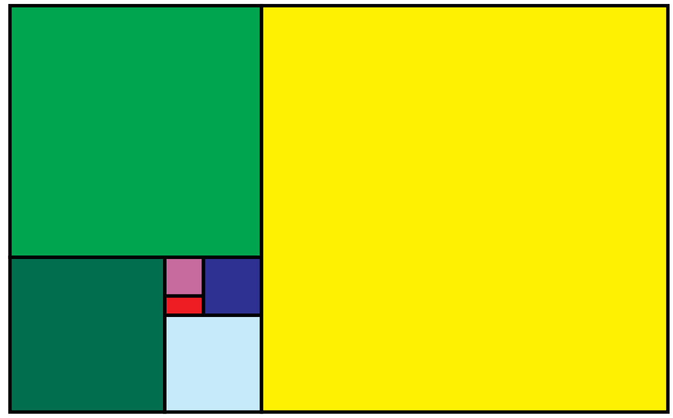
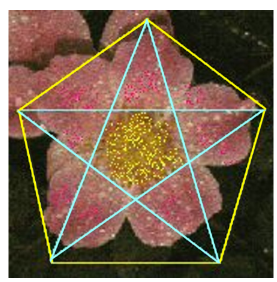


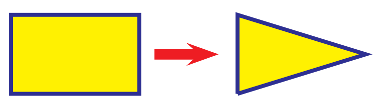
Figure 2

Here you have a rectangle made up of smaller and smaller squares arranged in spiral fashion. This is known as a **golden rectangle**.

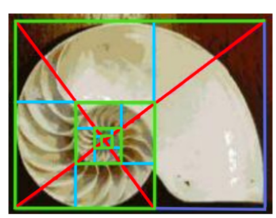
A rectangle is golden if it has the following special property: if you cut a square (shown in yellow) away from it, you are left with a rectangle of the same shape—so you can continue cutting off squares indefinitely.



A rose is a rose is a rose—and shares its pentagonal symmetry with many other flowers, from buttercups to petunias. The underlying **pentagram** (shown here in blue) is made up of **golden triangles,** each of which is obtained from a golden rectangle by collapsing one of the shorter sides.



This is the connection between golden rectangles and regular pentagons.



The *nautilus*, a tropical seashell, provides a beautiful example of a **logarithmic spiral**, described by a point revolving around a center while, at the same time, moving outward exponentially (like the tip of a clock-hand that grows as compound interest). Hence, it could just as well be called an exponential spiral. But people *love* the L-word—it sounds so impressive—and since it merely refers to the reverse way of looking at the same pattern, it is quite as legitimate as the E-word.

The scaffolding for such a curve is given by a kind of rectangular spiral (shown here in green) wrapping around

[†] Find more about the author and other interesting articles at http://www.math.ubc.ca/~hoek/Teaching/teaching.html.

a rectangular cross (in red).

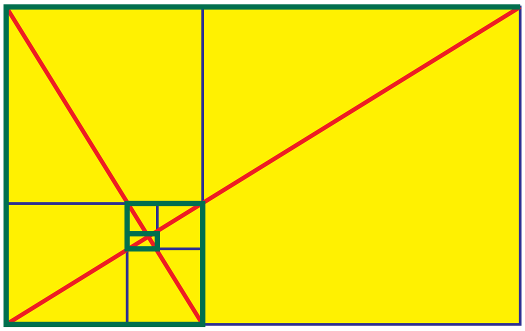


Figure 3

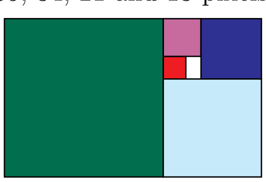
Figure 3 reveals an analogous structure within the golden rectangle. In fact, if you start with *any* non-square rectangle, draw a diagonal and (from another corner) a perpendicular to it, and then wrap a rectangular spiral around the resulting cross, you always get the framework for some logarithmic spiral. Doing this with a *golden* rectangle, gives you a bonus: your spiral can now be approximated by a bunch of (easily drawn) quarter circles. Alas, only *approximated*! The **golden spiral** shown in Figure 1 is not truly logarithmic—it is not even *smooth*: if you were driving along it with constant speed, you would feel a sudden jerk in the steering wheel as you changed circles in passing from the yellow square to the green one.

So much for the geometric inhabitants of this zoo—but what about Fibonacci and his numbers

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, . . . ?

This is the hub of the mystery, although it looks innocent and even playful at first sight. Through this sequence, Fibonacci (about 1200 AD) was attempting to model the growth of a population in which the newborn have to “sit out” one mating season before getting involved in the game. He actually thought of rabbits! What does this rabbit sequence have to do with the “divine proportion?”

The truth is that the squares shown in Figure 2 are not in an ideal plane, but are actually made up of coloured dots (called “pixels”), and have side lengths of 144, 89, 55, 34, 21 and 13 pixels, respectively—rabbit numbers!

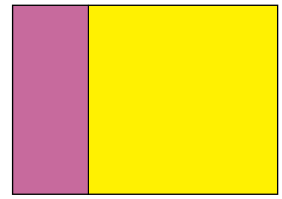


The red rectangle stuck in the middle of Figure 2 measures 13-by-8 pixels, and when enlarged 10-fold looks like what you see on the left.

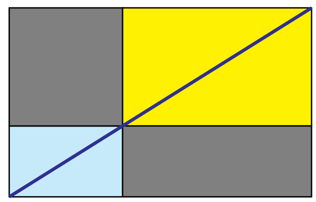
If you continue to cut off squares, you will—after 5 steps—wind up with the white square. And then? In a truly golden rectangle, you would be able to continue cutting off squares indefinitely. So, this is a fake—but a close imitation! This is no coincidence, as we shall now see.

Inflating Away the Imbalance

Inflating a rectangle means adding a square to its longer side, as shown on the right. The original rectangle is purple, while the added square is yellow. The enlarged rectangle will be referred to as the *inflation* of the old one. This is not standard terminology; it is an *ad hoc* term coined for the sole purpose of making this text easier to read. (Mathematics allows that kind of poetic license, as long as it is consistent.) We want to study the way inflation affects the *shape* of a rectangle. If it remains unaltered, we say the rectangle is *golden*.

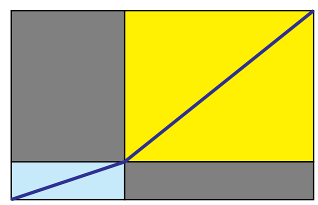


We shall eventually see that **continued inflation leads to a more and more golden shape**. But what does that mean? How can we compare the shapes of rectangles and check that one is “more golden” than another?

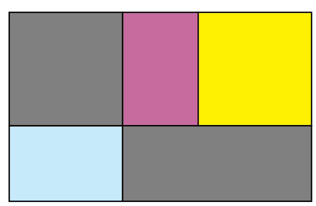


Of course, we all know that rectangles come in various shapes, from squat squares to the thinnest of strips. Here is a way to compare them: when two rectangles have the same shape—like the blue and the yellow ones depicted on the left—their diagonals line up as shown, splitting the whole diagram exactly in two. The two grey rectangles must therefore be equal in area.

If the blue and yellow shapes are not the same, their diagonals do not line up, and the complementary grey areas are not equal. This is the situation shown below on the right.

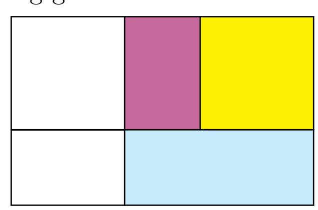


To compare the shape of a rectangle with the shape of its inflation, we therefore have to look at the diagram below, where the blue and purple rectangles are congruent (i.e., their corresponding sides are equal in length). The difference between the areas of the upper grey square and the lower grey strip will be called the *imbalance* of the purple rectangle.

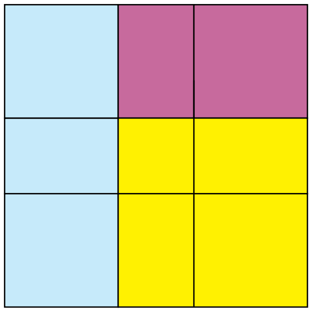


This is another *ad hoc* term intended to simplify the language. In some sense, the imbalance of a rectangle measures how far it is from being golden. The imbalance is zero in a golden rectangle.

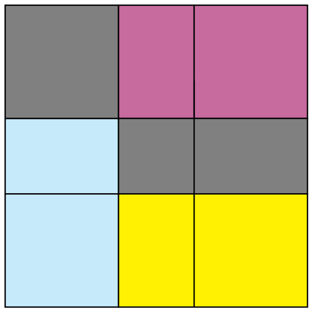
The grey square and strip in the diagram above, show up again—coloured yellow and blue, respectively—in the diagram at the right. (Make sure you can argue this in detail!).



This suggests another way of defining the imbalance of the purple rectangle: *It is the difference between the areas of (a) the square that must be added to inflate it and (b) the strip that must be added to the inflation to complete the larger square.*



Now let us look at the imbalance of the inflation, shown all in purple on the left. The square used to inflate the inflation is yellow, the strip required to complete the larger square is blue. What is the difference between the areas of these two shapes? The answer is shown on the right: since the blue and yellow rectangles are congruent, they contribute nothing to the new imbalance. It is still based on the difference between the two grey areas—just like the old one!



Conclusion: **Inflation does not change the size of the imbalance.** In particular, inflation keeps golden rectangles golden. Even better—since continued inflation keeps pumping up the area without changing the imbalance, the latter fades into insignificance compared with the size of the rectangles. More concretely, in the last four diagrams above, the imbalance happens to be five (square) pixels—a puny amount if we keep inflating the rectangles to wall size. In the end, our rectangles will—for all practical purposes—be golden.

Back to the Rabbits

So, where is the promised insight into the Fibonacci numbers? Well, if you attach a square to the longer side of a one-by-two rectangle, you get a two-by-three rectangle; further inflation yields a three-by-five rectangle, and so on, generating the sequence

1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, . . . ,

already seen on the previous page. Comparing the square of any of these numbers with the product of its two neighbours, you will always get a difference of 1. That is the imbalance. If, however, you begin with a one-by-three rectangle, continued inflation will spawn the Lucas sequence,

1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, . . . ,

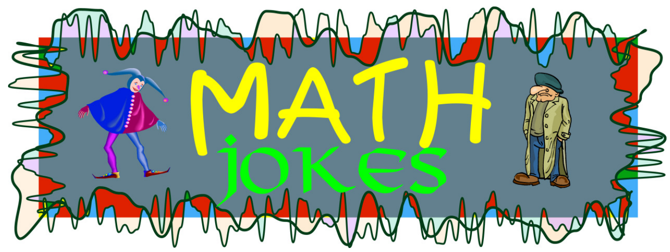
named after the French number wizard Edouard Lucas, who lived in the 19th century. Its imbalance is 5, and, of course, it too marches toward the divine proportion.

Algebraically, the preservation of the imbalance is very easy to see. After all, we are dealing here with number sequences

$$a,\ b,\ c,\ d,\ \ldots \text{ with } c = a + b \text{ and } d = b + c = a + 2b.$$

The imbalance for a, b, c is $b^2 - ac = b^2 - a(a + b)$. For b, c, d, it is $c^2 - bd = (a + b)^2 - b(a + 2b)$, which comes to $ac - b^2$, as you can (and should) easily check. In other words, its size remains constant but it changes from plus to minus and vice versa. Can you see this switching of signs in the pictorial proof above? How would you compare this algebraic proof with the geometric one? Would the conclusions of this comparison hold in every case where both algebra and geometry can be used? Try to come up with examples. . . .

What about the perfectly golden case, with zero imbalance: $b^2 = a(a + b)$? Is this possible with integers a and b? Remember: an $a - by - b$ rectangle would be golden, and could be "deflated" (as in Figure 2 on the previous page) ad infinitum, always keeping integer values for its sides. Could that be?. . . TO BE CONTINUED. . . .



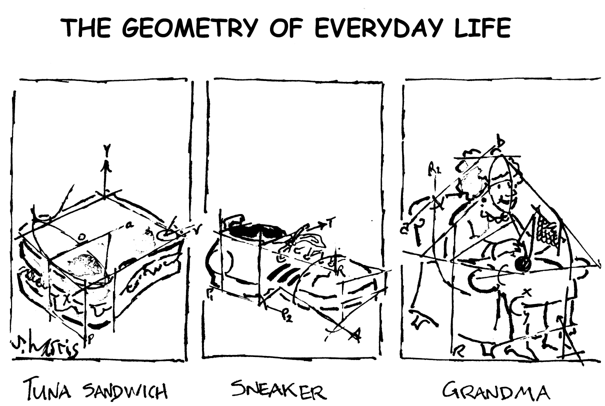
A mathematician gives a talk intended for a general audience. The talk is announced in the local newspaper, but he expects few people to show up because anyone who is not a mathematician will be unable to make any sense of the title: *Convex Sets and Inequalities*.

To his surprise, the auditorium is full when his talk begins. After he has finished, someone in the audience raises a hand.

"But you said nothing about the actual topic of your talk!"

"What topic do you mean?"

"Well, the one that was announced in the paper: *Convicts, Sex, and Inequality*."



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Sidney Harris

Q: What does the little mermaid wear?

A: An algae-bra.

One day, Jesus said to his disciples, "The Kingdom of Heaven is like $3x$ squared plus $8x$ minus 9."

A man who had just joined the disciples looked very confused, and asked Peter, "What on Earth, does he mean by that?"

Peter replied, "Don't worry—it's just another one of his parabolas."

Three Easy Tricks

by Ted Lewis†

These three math tricks will baffle most people. In each case, I will describe how you will present the trick to "Jessica and Jonathan," two of your spectators. The challenge for you is to figure out how it is done before peeking at the explanations at the end of the article. Each trick uses a different but simple mathematical principle.

Finding a Card

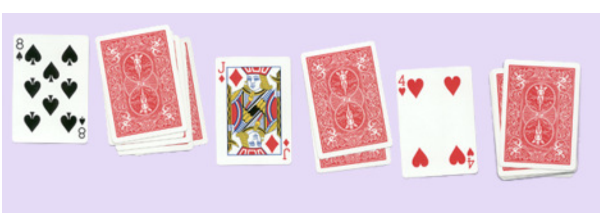
You hand a deck of cards to Jessica. Ask her to think of a number no smaller than 11 and no larger than 19, and to deal that many cards face down in a pile on the table. Turn your back so you cannot see how many cards she deals, and tell her to do it silently so that you cannot count the cards.

When she has finished, take the rest of the deck from her and place it alongside the small pile she has dealt. Ask her to pick up the small pile of cards that she dealt, add the two digits of her chosen number, then deal that many cards face down onto the top of the deck. So, if the number she chose was 17, she would now deal eight cards onto the top of the deck. To make the cards dealt completely random, ask her to shuffle the cards she is left holding, and to remember the card that is on the bottom without showing it to you.

Then have her put this little packet of cards face down on top the deck. The card she looked at is now sandwiched somewhere in the middle the deck.

Now, have her cut the deck twice. Although there seems to be no way for you to do so, you look through the deck and immediately display her card.

A Prediction



†Ted Lewis is a professor in the Department of Mathematical Sciences at the University of Alberta. His web site is http://www.math.ualberta.ca/~tlewis.

A sealed envelope is placed on the table.

You fan the cards and let Jonathan freely choose one and place it face up on the table. You may even let him look at the faces of the cards as you fan them. Let us suppose that card is the 8♠. You hand the deck to Jonathan, face down, and ask him to deal some cards face down next to the 8♠, counting from the value of the card up to the King. In this case, he would he would count, "Nine, 10, Jack, Queen, King" as he deals five face down cards. Ask him to turn the next card face up and repeat the process. Say it is the J♢: Jonathan deals two face down cards as he counts, "Queen, King". He turns up the next card, say the 4♡, counting, "Five, six, . . ., Jack, Queen, King" as he deals the face down cards. Now, he totals the face up cards: $8 + 11 + 4 = 23$, and he counts 23 more cards face down on the table.

He places the next card face up the table; it is, say, the ace of hearts. He opens the envelope and finds a paper inside with the message, "You will choose the ace of hearts!"

In this trick, the cards that Jonathan turns face up do not have to be an eight, a Jack and a four—the trick works no matter what they are.

Two Spectators

Both Jessica and Jonathan participate in this trick. Three cards are placed face down on the table. Jessica is going to do some calculations, and using a calculator would be a good idea. Jonathan will not need to do any calculations.

Ask Jessica to write down any three-digit number. You turn your back so that you cannot see the number, and you ask Jessica to create a six-digit number by writing the number alongside itself. "For example," you explain, "If the original number was 123, the six-digit number would be 123123."

Ask Jonathan to choose one of the three cards, and turn it face up. Then ask Jessica to divide the six-digit number by the value of the face up card, ignoring any remainder, and to write down the result and circle it. A Jack counts as 11, a Queen as 12, and a King as 13. You may mention that, "Although I don't have any idea what your result is, I have a very strong feeling that there was no remainder."

Ask Jonathan to choose one of the two remaining face down cards and turn it face up. Ask Jessica to write down the answer when the circled result is divided by the value of the card. Surprisingly, again there is no remainder.

Ask Jessica to divide the latest result by the original three-digit number, and to write down the result of this calculation. This is the final result, and incredibly, there is still no remainder.

Now review the situation for Jessica and Jonathan—Jessica was free to choose *any three-digit number* whatsoever. Jonathan was free to choose *any two* of the three face down cards. Would it not be quite a miracle if the final result happened to be the value of the remaining face down card? But that is exactly what happens.

There is some chicanery involved here—the three cards are not arbitrary. Can you figure out what they are? If you are good at arithmetic, you should be able to do so.

How the Tricks Work

Finding a Card

Before handing the cards to Jessica, memorize the card that is tenth from the top. This is the locator card, and her chosen card will always be the one directly above it.

There is an interesting mathematical reason why this works. A positive integer and the sum of its digits have the same remainder when you divide by nine (see the box at the end of this article). This means that if you have multi-digit number and from it you subtract the sum of those digits, the result must be a number that is divisible by nine, and it has to be exactly nine if the multi-digit number is between 10 and 19. Now recall what Jessica does: she deals $1n$ cards on the table, where $1n$ is one of the numbers $11, 12, 13, \ldots, 19$. This reverses the order of the $1n$ cards. Then she picks up these cards and deals $(1 + n)$ of them onto the top of the deck. Since the cards are in reverse order, she is left holding the original top nine cards of the deck, and the tenth card—the locator card—is now back in position on top of the deck. Jessica shuffles the nine cards that she is holding, remembers the bottom card, and puts everything on top of the deck. So, the chosen card ends up directly on top of the locator card. Now I think that's rather neat.

A Prediction

Put the force card (the A♡) as the 43rd card from the top of the deck. The counting method ensures that Jonathan will put 42 cards onto the table. The chart below shows why:

Face up card	Number of face down cards
8♠	13 − 8
J♢	13 − 11
4♡	13 − 4

The total number of cards on the table at this point is $3+(13-8)+(13-11)+(13-4)$. Jonathan then proceeds to deal $(8 + 11 + 4)$ more cards, for a total of $3 + 3(13)$ cards. Incidentally, when doing this trick, make sure that Jonathan chooses his first card from the part of the deck above the 43rd card. Do this by slowly fanning out only the top half of the deck for him to choose from.

Two Spectators

The three cards that are placed face down must be a seven, a Jack, and a King. The product of the values of the three cards is 1001, and the product of the three-digit number xyz with 1001 is $xyzxyz$. In other words, $xyzxyz = xyz \cdot 7 \cdot 11 \cdot 13$. By following your instructions, Jessica is dividing the six-digit number by three of its factors. What's left is the fourth factor—the value of the remaining face down card.

If you would like to learn more about magic that is based on simple mathematics, here are two excellent books:

Mathematical Magic by William Simon, preface by Martin Gardner, Dover Publications, New York, 1964.

Self-Working Card Tricks by Karl Fulves, Dover Publications, New York, 1976.

Simon's book explains the mathematics behind the tricks quite thoroughly. Fulves' book includes many tricks that are not mathematically based, and you will have to think a bit to understand the mathematics behind those that are.

The three tricks in this article are not from these books. Nevertheless, I would like to quote the advice given by Bill Simon in the forward to his book: "Be sure to run through the items you plan to use so that you can demonstrate them with certainty, with complete understanding, and in an entertaining manner. You will then be sure of performing real *mathematical magic!*"

Dividing by 9

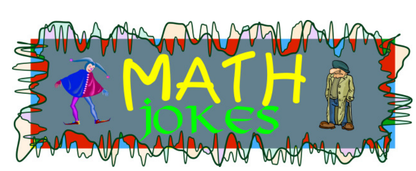
Suppose that xyz is a positive 3-digit number. This means that the number is

$$100x + 10y + z\,.$$

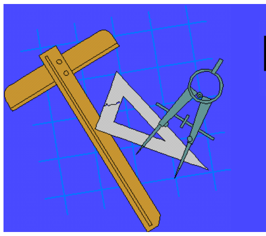
Now rearrange it:

$$(99x + 9y) + (x + y + z)\,.$$

So, when we divide the number xyz by 9, we get the same remainder as we do when we divide $x + y + z$ by 9. The same reasoning works regardless of the number of digits.



A logician at Safeway:
"Paper or plastic?"
"Not 'not paper and not plastic'!"



Math Strategies

Inequalities for Convex Functions (Part I)

by Dragos Hrimiuc[†]

1. Convex functions.

Convex functions are powerful tools for proving a large class of inequalities. They provide an elegant and unified treatment of the most important classical inequalities.

A real-valued function on an interval I is called *convex* if

$$f(\lambda x+(1-\lambda)y) \le \lambda f(x)+(1-\lambda)f(y) \qquad (1)$$

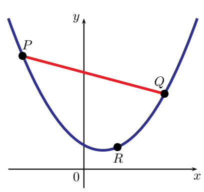
for every $x,\ y\in I$ and $\lambda\in[0,1]$; it is called *strictly convex* if

$$f(\lambda x+(1-\lambda)y) < \lambda f(x)+(1-\lambda)f(y) \qquad (2)$$

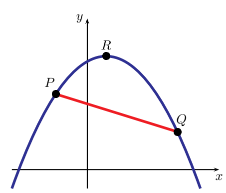
for every $x,y\in I,\quad x\neq y$ and $\lambda\in(0,1)$.

Notice: f is called *concave (strictly concave)* on I if $-f$ is *convex (strictly convex)* on I.

The geometrical meaning of convexity is clear: f is strictly convex if and only if for every two points $P=(x,f(x))$ and $Q=(y,f(y))$ on the graph of f, the point $R=(z,f(z))$ lies below the segment PQ for every z between x and y.



The graph of a convex function



The graph of a concave function

How to recognize a convex function without the graph? We can use (1) directly, but the following criterion is often very useful:

> **Test for Convexity:** Let f be a twice differentiable function on I. Then
> - f is convex on I if $f''(x)\ge 0$ for every $x\in I$.
> - f is strictly convex on I if $f''(x)>0$ for every x in the interior of I.

Remark: If f is a continuous function on I, then it can be proved that f is convex if and only if for all $x_1,\ x_2\in I$

$$f\left(\frac{x_1+x_2}{2}\right)\le\frac{f(x_1)+f(x_2)}{2};$$

and it is strictly convex if and only if

$$f\left(\frac{x_1+x_2}{2}\right)<\frac{f(x_1)+f(x_2)}{2}$$

for all $x_1,\ x_2\in I,\ x_1\neq x_2$ (see [1]).

Here are some basic examples of strictly convex functions:
(i) $f(x)=x^{2n},\quad x\in\mathbb{R}$ and n is a positive integer;
(ii) $f(x)=x^p,\ x\ge 0,\ p>1$;
(iii) $f(x)=\frac{1}{(x+a)^p},\ x>-a,\ p>0$;
(iv) $f(x)=\tan x,\ x\in\left[0,\frac{\pi}{2}\right)$;
(v) $f(x)=e^x,\ x\in\mathbb{R}$.

The following are examples of strictly concave functions:
(i) $f(x)=\sin x,\ x\in[0,\pi]$;
(ii) $f(x)=\cos x,\ x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$;
(iii) $f(x)=\ln x,\ x\in(0,\infty)$;
(iv) $f(x)=x^p,\ x\ge 0,\ p\in(0,1)$.

Notice:

1. The linear function $f(x)=ax+b,\ x\in R$ is convex and also concave.
2. The sum of two convex (concave) functions is a convex (concave) function.

2. Jensen's Inequality.

Jensen's inequality is an extension of (1). It was named after the Danish mathematician who proved it in 1905.

> **Jensen's Inequality:** Let $f: I\to\mathbb{R}$ be a convex function. Let $x_1,\ldots,x_n\in I$ and $\lambda_1,\ldots,\lambda_n\ge 0$ such that $\lambda_1+\lambda_2+\ldots+\lambda_n=1$. Then
>
> $$f(\lambda_1x_1+\lambda_2x_2\ldots+\lambda_nx_n)\le\lambda_1f(x_1)+\ldots+\lambda_nf(x_n). \qquad (3)$$

Proof: Let's use mathematical induction. The inequality is true for $n=1$. Now assume that it is true for $n=k$, and let's show that it remains true for $n=k+1$.

Let $x_1,\ldots,x_k,x_{k+1}\in I$ and let $\lambda_1,\ldots,\lambda_k,\lambda_{k+1}\ge 0$ with $\lambda_1+\lambda_2+\ldots+\lambda_k+\lambda_{k+1}=1$. At least one of $\lambda_1,\lambda_2,\ldots,\lambda_{k+1}$ must be less then 1 (otherwise the inequality is trivial). Without loss of generality, let $\lambda_{k+1}<1$ and $u=\frac{\lambda_1}{1-\lambda_{k+1}}x_1+\ldots+\frac{\lambda_k}{1-\lambda_{k+1}}x_k$. We have

$$\frac{\lambda_1}{1-\lambda_{k+1}}+\ldots+\frac{\lambda_k}{1-\lambda_{k+1}}=1,$$

and also

$$\lambda_1x_1+\ldots+\lambda_kx_k+\lambda_{k+1}x_{k+1}=(1-\lambda_{k+1})u+\lambda_{k+1}x_{k+1}.$$

Now, since f is convex,

$$f((1-\lambda_{k+1})u+\lambda_{k+1}x_{k+1})\le(1-\lambda_{k+1})f(u)+\lambda_{k+1}f(x_{k+1})$$

and, by our induction hypothesis,

$$f(u)\le\frac{\lambda_1}{1-\lambda_{k+1}}f(x_1)+\ldots+\frac{\lambda_k}{1-\lambda_{k+1}}f(x_k).$$

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Hence, combining the above two inequalities, we get:

$$f(\lambda_1 x_1 + \ldots + \lambda_{k+1}x_{k+1}) \leq \lambda_1 f(x_1) + \ldots + \lambda_{k+1} f(x_{k+1}).$$

Thus, the inequality is established for $n = k + 1$, and therefore, by mathematical induction, it holds for any positive integer n. □

Remarks:

1. For strictly convex functions, the inequality in ③ holds if and only if $x_1 = x_2 = \ldots = x_n$. Use mathematical induction to prove it.
2. If $\lambda_1 = \lambda_2 = \ldots = \lambda_n = \frac{1}{n}$, then ③ becomes
$$f\left(\frac{x_1 + \ldots + x_n}{n}\right) \leq \frac{f(x_1) + \ldots + f(x_n)}{n}. \qquad ④$$
3. If f is a concave function, then ③ and ④ read as
$$f(\lambda_1 x_1 + \ldots + \lambda_n x_n) \geq \lambda_1 f(x_1) + \ldots + \lambda_n f(x_n), \qquad ③'$$
and
$$f\left(\frac{x_1 + \ldots + x_n}{n}\right) \geq \frac{f(x_1) + \ldots + f(x_n)}{n}. \qquad ④'$$

Jensen's Inequality has variety of applications. It can be used to prove many of the most important classical inequalities.

Weighted AM–GM Inequality:
Let $x_1, \ldots, x_n \geq 0$, $\lambda_1, \ldots, \lambda_n > 0$ such that $\lambda_1 + \ldots + \lambda_n = 1$. Then

$$\lambda_1 x_1 + \ldots + \lambda_n x_n \geq x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_n^{\lambda_n} \qquad ⑤$$

The equality holds if and only if $x_1 = x_2 = \ldots = x_n$.

Proof: We may assume that $x_1, \ldots, x_n > 0$. Let $f(x) = \ln x$, $x \in (0, \infty)$. Since f is strictly concave on $(0, \infty)$, by using ③′ we get:

$$\ln(\lambda_1 x_1 + \ldots + \lambda_n x_n) \geq \lambda_1 \ln x_1 + \ldots + \lambda_n \ln x_n,$$

or, equivalently, $\ln(\lambda_1 x_1 + \ldots + \lambda_n x_n) \geq \ln x_1^{\lambda_1} \ldots x_n^{\lambda_n}$, and hence,

$$\lambda_1 x_1 + \ldots + \lambda_n x_n \geq x_1^{\lambda_1} \ldots x_n^{\lambda_n}$$

(since $f(x) = \ln x$ is a strictly increasing function). □

By taking $\lambda_1 = \lambda_2 = \ldots = \lambda_n = \frac{1}{n}$ in ⑤, we obtain:

AM–GM Inequality:
If $x_1, \ldots, x_n \geq 0$, then

$$\frac{x_1 + x_2 + \ldots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n}, \qquad ⑥$$

with equality if and only if $x_1 = x_2 = \ldots = x_n$.

Let $x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n > 0$ be such that $\lambda_1 + \ldots + \lambda_n = 1$. For each $t \in \mathbb{R}$, $t \neq 0$, the weighted mean M_t of order t is defined as

$$M_t = \left(\frac{\lambda_1 x_1^t + \lambda_2 x_2^t + \ldots + \lambda_n x_n^t}{n}\right)^{\frac{1}{t}}.$$

Some particular situations are significant:

$$M_1 = \frac{\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n}{n}$$

is called the *weighted arithmetic mean* (*WAM*); and

$$M_{-1} = \frac{n}{\frac{\lambda_1}{x_1} + \frac{\lambda_2}{x_2} + \ldots + \frac{\lambda_n}{x_n}}$$

is called the *weighted harmonic mean* (*WHM*),

$$M_2 = \sqrt{\frac{\lambda_1 x_1^2 + \lambda_2 x_2^2 + \ldots + \lambda_n x_n^2}{n}}$$

is called the *weighted root mean square (WRMS)*.

It can be shown by using l'Hôpital's Rule that

$$\lim_{t \to 0} M_t = x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_n^{\lambda_n}.$$

So, if we denote $M_0 = \lim_{t \to 0} M_t$, we see that

$$M_0 = x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_n^{\lambda_n},$$

which is called the *weighted geometric mean* (*WGM*).
Also, if we set

$$M_\infty = \lim_{t \to \infty} M_t \quad \text{and} \quad M_{-\infty} = \lim_{t \to -\infty} M_t,$$

we obtain

$$M_\infty = \max\{x_1, \ldots, x_n\}, \qquad M_{-\infty} = \min\{x_1, \ldots, x_n\}.$$

Power Mean Inequality:
Let $x_1, x_2, \ldots, x_n, \lambda_1, \ldots, \lambda_n > 0$ be such that $\lambda_1 + \ldots + \lambda_n = 1$. If t and s are non-zero real numbers such that $s < t$, then

$$\left(\frac{\lambda_1 x_1^s + \ldots + \lambda_n x_n^s}{n}\right)^{\frac{1}{s}} \leq \left(\frac{\lambda_1 x_1^t + \ldots + \lambda_n x_n^t}{n}\right)^{\frac{1}{t}}. \qquad ⑦$$

Proof: If $0 < s < t$ or $s < 0 < t$, the inequality ⑦ is obtained by applying Jensen's Inequality ③ to the strictly convex function $f(x) = x^{\frac{t}{s}}$. Indeed, if $a_1, a_2, \ldots, a_n, \lambda_1, \ldots, \lambda_n > 0$ and $\lambda_1 + \ldots + \lambda_n = 1$, then

$$\left[\frac{\lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_n a_n}{n}\right]^{\frac{t}{s}} \leq \frac{\lambda_1 a_1^{\frac{t}{s}} + \lambda_2 a^{\frac{t}{s}} + \ldots + \lambda_n a_n^{\frac{t}{s}}}{n}.$$

By choosing $a_1 = x_1^s, \ldots, a_n = x_n^s$, we immediately obtain ③.

If $s < t < 0$, then $0 < -t < -s$, and by applying ⑦ for $\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}$, we get

$$\left[\frac{\lambda_1(\frac{1}{x_1})^{-t} + \ldots + \lambda_n(\frac{1}{x_n})^{-t}}{n}\right]^{\frac{1}{-t}} \leq \left[\frac{\lambda_1(\frac{1}{x_1})^{-s} + \ldots + \lambda_n(\frac{1}{x_n})^{-s}}{n}\right]^{\frac{1}{-s}},$$

which can be rewritten as ⑦. □

Remark: If $t < 0 < s$, then $M_t \leq M_0 \leq M_s$. Also, we have the following classical inequality:

$$M_{-\infty} \leq M_{-1} \leq M_0 \leq M_1 \leq M_2 \leq M_\infty.$$

Hölder's Inequality: If $p, q > 1$ are real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$ and $a_1, \ldots, a_n, b_1, \ldots, b_n$ are real (complex) numbers, then

$$\sum_{k=1}^n |a_k||b_k| \leq \left(\sum_{k=1}^n |a_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^n |b_k|^q\right)^{\frac{1}{q}}. \quad ⑧$$

Proof: We may assume that $|a_k| > 0$, $k = 1, \ldots, n$. (Why?) The function $f(x) = x^q$ is strictly convex on $(0, \infty)$, hence, by Jensen's Inequality,

$$\left(\sum_{k=1}^n \lambda_k x_k\right)^q \leq \sum_{k=1}^n \lambda_k x_k^q,$$

where $x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_n > 0$ and $\lambda_1 + \ldots + \lambda_n = 1$. Let $A = \sum_{k=1}^n |a_k|^p$. By choosing $\lambda_k = \frac{1}{A}|a_k|^p$ and $x_k = \frac{1}{\lambda_k}|a_k||b_k|$ in the above inequality, we obtain ⑧. □

Remarks:

1. In ⑧, the equality holds if and only if $x_1 = x_2 = \ldots = x_n$. That is,

$$\frac{|a_1|^p}{|b_1|^q} = \frac{|a_2|^p}{|b_2|^q} = \ldots = \frac{|a_n|^p}{|b_n|^q}.$$

Notice that this chain of equalities is taught in the following way: if a certain $b_k = 0$, then we should have $a_k = 0$.

2. If $p = q = 2$, Hölder's Inequality is just **Cauchy's Inequality**:

$$\left(\sum_{k=1}^n |a_k||b_k|\right)^2 \leq \left(\sum_{k=1}^n |a_k|^2\right)\left(\sum_{k=1}^n |b_k|^2\right).$$

The equality occurs when

$$\frac{|a_1|}{|b_1|} = \frac{|a_2|}{|b_2|} = \ldots = \frac{|a_n|}{|b_n|}.$$

Minkowski's Triangle Inequality: If $p > 1$ and $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \geq 0$, then

$$\left(\sum_{k=1}^n (a_k + b_k)^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^n a_k^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^n b_k^p\right)^{\frac{1}{p}}. \quad ⑨$$

Proof: We may assume $a_k > 0$, $k = 1, \ldots, n$. (Why?) The function $f(x) = \left(1 + x^{\frac{1}{p}}\right)^p$, $x \in (0, \infty)$, is strictly concave since $f''(x) = \frac{1-p}{p}\left(1 + x^{\frac{1}{p}}\right)^{p-2} \cdot x^{\frac{1}{p}-2} < 0$. By Jensen's Inequality,

$$\left[1 + \left(\sum_{k=1}^n \lambda_k x_k\right)^{\frac{1}{p}}\right]^p \geq \sum_{k=1}^n \lambda_k \left(1 + x_k^{\frac{1}{p}}\right)^p,$$

where $x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_n > 0$ and $\lambda_1 + \ldots + \lambda_n = 1$. Let $A = \sum_{k=1}^n a_k^p$. By taking $\lambda_k = \frac{a_k^p}{A}$ and $x_k = \frac{b_k^p}{a_k^p}$ for $k = 1, \ldots, n$ in the above inequality, we obtain ⑨.

Remarks.

1. The equality in ⑨ occurs if and only if

$$\frac{b_1}{a_1} = \frac{b_2}{a_2} = \ldots = \frac{b_n}{a_n}.$$

2. If $p = 2$ we get the so-called Triangle Inequality:

$$\sqrt{\sum_{k=1}^n (a_k + b_k)^2} \leq \sqrt{\sum_{k=1}^n a_k^2} + \sqrt{\sum_{k=1}^n b_k^2}.$$

Example 1. If $a, b \geq 0$ and $a + b = 2$, then

$$\left(1 + \sqrt[5]{a}\right)^5 + (1 + \sqrt[5]{b})^5 \leq 2^6.$$

Solution: Since $f(x) = (1 + \sqrt[5]{x})^5$ is strictly concave on $[0, \infty)$, by using Jensen's Inequality ④ we get

$$2\left(1 + \sqrt[5]{\frac{a+b}{2}}\right)^5 \geq (1 + \sqrt[5]{a})^5 + (1 + \sqrt[5]{b})^5.$$

By substituting $a + b = 2$, we get the required inequality. The equality occurs when $a = b = 1$. □

Example 2. If $a, b, c > 0$, then

$$a^a \cdot b^b \cdot c^c \geq \left(\frac{a+b+c}{3}\right)^{a+b+c}.$$

Solution: The above inequality is equivalent to

$$\ln(a^a \cdot b^b \cdot c^c) \geq \ln\left(\frac{a+b+c}{3}\right)^{a+b+c},$$

or

$$a \ln a + b \ln b + c \ln c \geq (a+b+c) \ln \left(\frac{a+b+c}{3} \right).$$

Let $f(x) = x \ln x$, $x \in (0, \infty)$. Since $f''(x) = \frac{1}{x} > 0$, the function f is strictly convex on $(0, \infty)$. Now, the above inequality follows from (4). □

Example 3. If a, b, $c > 0$ then

$$\frac{a}{a+3b+3c} + \frac{b}{3a+b+3c} + \frac{c}{3a+3b+c} \geq \frac{3}{7}.$$

Solution: Let s be a positive number and $f(x) = \frac{x}{s-x} = \frac{s}{s-x} - 1$, $x \in (0, s)$. The function f is strictly convex since $f''(x) = \frac{2s}{(s-x)^3} > 0$. We get:

$$\frac{2a}{s-2a} + \frac{2b}{s-2b} + \frac{2c}{s-2c} \geq 3 \frac{\frac{1}{3}(2a+2b+2c)}{s - \frac{1}{3}(2a+2b+2c)},$$

or

$$\frac{a}{s-2a} + \frac{b}{s-2b} + \frac{c}{s-2c} \geq \frac{3(a+b+c)}{3s - 2(a+b+c)}.$$

If we take $s = 3(a+b+c)$, the required inequality follows. □

Example 4. If $a_1, a_2, \dots, a_n \geq 1$, then

$$\sum_{k=1}^{n} \frac{1}{1+a_k} \geq \frac{n}{1+\sqrt[n]{a_1 a_2 \dots a_n}}.$$

Solution: Let $f(x) = \frac{1}{1+e^x}$, $x \in [0, \infty)$. The function f is strictly convex since $f''(x) = \frac{e^x(e^x-1)}{(e^x+1)^3} > 0$ on $(0, \infty)$. Using (4), we get

$$\sum_{k=1}^{n} \frac{1}{1+e^{x_k}} \geq \frac{n}{1+e^{\frac{1}{n}\sum_{k=1}^{n} x_k}}.$$

By taking $x_k = \ln a_k$, $k = 1, \dots, n$, we obtain the required inequality. □

Example 5. For a triangle with angles α, β and γ, the following inequalities hold:

- $\sin\alpha + \sin\beta + \sin\gamma \leq \frac{3\sqrt{3}}{2}$;
- $\sqrt{\sin\alpha} + \sqrt{\sin\beta} + \sqrt{\sin\gamma} \leq 3\sqrt[4]{\frac{3}{4}}$;
- $\sin\alpha \cdot \sin\beta \cdot \sin\gamma \leq \frac{3\sqrt{3}}{8}$;
- $\cos\alpha \cdot \cos\beta \cdot \cos\gamma \leq \frac{1}{8}$;
- $\sec\frac{\alpha}{2} + \sec\frac{\beta}{2} + \sec\frac{\gamma}{2} \geq 2\sqrt{3}$.

Solution: Use the Jensen Inequality for the strictly concave functions $\sin x$, $\sqrt{\sin x}$, $\ln \sin x$, $\ln \cos x$, and for the strictly convex function $\sec\frac{x}{2}$, $x \in (0, \pi)$. □

Example 6. Let $a_1, \dots, a_n, \lambda_1, \dots, \lambda_n > 0$ and $\lambda_1 + \dots + \lambda_n = 1$. If $a_1^{\lambda_1} \dots a_n^{\lambda_n} = 1$, then

$$a_1 + a_2 + \dots + a_n \geq \frac{1}{\lambda_1^{\lambda_1} \dots \lambda_n^{\lambda_n}}.$$

The equality occurs if and only if $a_k = \frac{\lambda_k}{\lambda_1^{\lambda_1} \dots \lambda_n^{\lambda_n}}$ for $k = 1, \dots, n$.

Solution: By using the Weighted AM–GM Inequality, we have

$$\begin{aligned} a_1 + \dots + a_n &= \lambda_1 \left(\tfrac{a_1}{\lambda_1}\right) + \dots + \lambda_n \left(\tfrac{a_n}{\lambda_n}\right) \\ &\geq \left(\tfrac{a_1}{\lambda_1}\right)^{\lambda_1} \dots \left(\tfrac{a_n}{\lambda_n}\right)^{\lambda_n} \\ &= \tfrac{1}{\lambda_1^{\lambda_1} \dots \lambda_n^{\lambda_n}}. \end{aligned}$$

The equality occurs if and only if

$$\frac{a_1}{\lambda_1} = \frac{a_2}{\lambda_2} = \dots = \frac{a_n}{\lambda_n}.$$

in which case the constraint $a_1^{\lambda_1} \dots a_n^{\lambda_n} = 1$ leads to $a_k = \frac{\lambda_k}{\lambda_1^{\lambda_1} \dots \lambda_n^{\lambda_n}}$ for $k = 1, \dots, n$. □

Example 7. If $a_1, \dots, a_n > 0$ and $a_1 a_2 \dots a_n = 1$, then

$$a_1 + \sqrt{a_2} + \dots + \sqrt[n]{a_n} \geq \frac{n+1}{2}.$$

Hint: Use the Weighted AM–GM Inequality.

Example 8.
(i) If a, b, $c > 0$, then

$$\frac{a^{10} + b^{10} + c^{10}}{a^5 + b^5 + c^5} \geq \left(\frac{a+b+c}{3}\right)^5.$$

(ii) If $a_1, a_2, \dots, a_n > 0$ and $k > p \geq 0$, then

$$\frac{a_1^k + \dots + a_n^k}{a_1^p + \dots + a_n^p} \geq \left(\frac{a_1 + \dots + a_n}{n}\right)^{k-p}.$$

Solution: (i) A particular case of (ii).

(ii) Let $M_t = \left(\frac{a_1^t + a_1^t + \dots + a_n^t}{n}\right)^{\frac{1}{t}}$. Then, by using the Power Mean Inequality (7), we get

$$\begin{aligned} a_1^k + a_2^k + \dots + a_n^k &= n M_k^k = n M_k^p M_k^{k-p} \\ &\geq n M_p^p M_1^{k-p} \quad \text{by (7)} \\ &= (a_1^p + \dots + a_n^p) \cdot \left(\tfrac{a_1 + \dots + a_n}{n}\right)^{k-p}. \end{aligned}$$

□

Example 9. If $a_1, a_2, \ldots, a_n > 0$, then

(i) $a_1^{n+1} + \ldots + a_n^{n+1} \geq a_1 \ldots a_n(a_1 + \ldots + a_n)$,

(ii) $a_1^{n-1} + \ldots + a_n^{n-1} \geq a_1 \ldots a_n \left(\frac{1}{a_1} + \ldots + \frac{1}{a_n}\right)$.

Solution: (i)

$$\begin{aligned} a_1^{n+1} + \ldots + a_n^{n+1} &= nM_{n+1}^{n+1} = nM_{n+1}^n M_{n+1}^1 \\ &\geq nM_0^n M_1^1 = a_1 \ldots a_n(a_1 + \ldots + a_n). \end{aligned}$$

(ii) See (i). □

Example 10. If $a, b, c, x, y, z, n > 0$, and

$$(a^n + b^n + c^n)^{n+1} = x^n + y^n + z^n,$$

then

$$\frac{a^{n+1}}{x} + \frac{b^{n+1}}{y} + \frac{c^{n+1}}{z} \geq 1.$$

Solution:

$$\begin{aligned} a^n + b^n + c^n &= \left[\frac{a^{n+1}}{x}\right]^{\frac{n}{n+1}} \cdot x^{\frac{n}{n+1}} + \left[\frac{b^{n+1}}{y}\right]^{\frac{n}{n+1}} \cdot y^{\frac{n}{n+1}} \\ &\quad + \left[\frac{c^{n+1}}{z}\right]^{\frac{n}{n+1}} \cdot z^{\frac{n}{n+1}}. \end{aligned}$$

By using Hölder's Inequality with $p = \frac{n+1}{n}$ and $q = \frac{p}{p-1} = n+1$, we obtain:

$$a^n+b^n+c^n \leq \left(\frac{a^{n+1}}{x} + \frac{b^{n+1}}{y} + \frac{c^{n+1}}{z}\right)^{\frac{n}{n+1}} (x^n+y^n+z^n)^{\frac{1}{n+1}},$$

and the required inequality follows. □

Example 11. If $a_1, \ldots, a_n, b_1, \ldots, b_n > 0$, then

$$\frac{(a_1 + \ldots + a_n)^{n+1}}{(b_1 + \ldots + b_n)^n} \leq \frac{a_1^{n+1}}{b_1^n} + \ldots + \frac{a_n^{n+1}}{b_n^n}.$$

Solution: By using Hölder's Inequality with $p = n+1$, $q = \frac{n+1}{n}$, we get

$$\begin{aligned} a_1 + \ldots + a_n &= \left[\frac{a_1^{n+1}}{b_1^n}\right]^{\frac{1}{n+1}} \cdot b_1^{\frac{n}{n+1}} + \ldots + \left[\frac{a_n^{n+1}}{b_n^n}\right]^{\frac{1}{n+1}} \cdot b_n^{\frac{n}{n+1}} \\ &\leq \left[\frac{a_1^{n+1}}{b_1^n} + \ldots + \frac{a_n^{n+1}}{b_n^n}\right]^{\frac{1}{n+1}} (b_1 + \ldots + b_n)^{\frac{n}{n+1}}, \end{aligned}$$

so

$$(a_1+\ldots+a_n)^{n+1} \leq \left(\frac{a_1^{n+1}}{b_1^n} + \ldots + \frac{a_n^{n+1}}{b_n^n}\right)(b_1+\ldots+b_n)^n,$$

from which we get the required inequality. □

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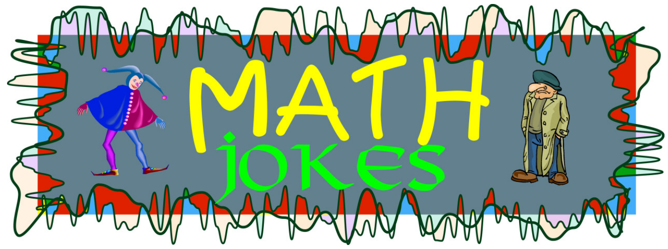
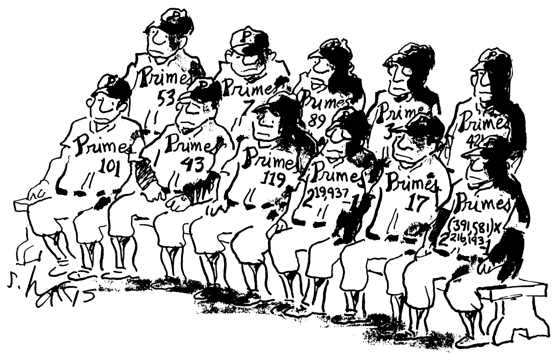


"Divide fourteen sugar cubes into three cups of coffee so that each cup has an odd number of sugar cubes."

"One, one and twelve."

"But twelve isn't odd!"

"It's an odd number of cubes to put in a cup of coffee. . . ."



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A mathematical biologist spends his vacation hiking in the Scottish highlands. One day, he encounters a shepherd with a large herd. One of these cuddly, woolly sheep would make a great pet, he thinks. . .

"How much for one of your sheep?" he asks the shepherd.

"They aren't for sale," the shepherd replies.

The math biologist ponders for a moment and then says, "I will give you the precise number of sheep in your herd without counting. If I'm right, don't you think that I deserve one of them as a reward?"

The shepherd nods.

The math biologist says, "387."

The shepherd is silent for a while, and then says, "You're right. I hate to lose any of my sheep, but I promised—one of them is yours. Take your pick!"

The math biologist grabs one of the animals, puts it on his shoulders, and is about to march on, when the shepherd says, "Wait! I will tell you what your profession is, and if I'm right, I'll get the animal back."

"That's fair enough."

"You must be a mathematical biologist."

The man is stunned. "You're right. But how could you tell?"

"That's easy! You gave me the precise number of sheep without counting—and then you picked my dog. . . ."

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A group of mathematicians and a group of engineers are traveling together by train to attend a conference on mathematical methods in engineering. Each engineer has a ticket, whereas only one of the mathematicians has one. Of course, the engineers laugh at the unworldly mathematicians and look forward to the moment when the conductor arrives.

Suddenly, one of the mathematicians shouts, "Conductor coming!"

All of the mathematicians disappear into one washroom. The conductor checks the ticket of each engineer and then knocks on the washroom door, "Your ticket, please."

The mathematicians stick the one ticket they have under the door, the conductor checks it and leaves. A few minutes later, when it is safe, the mathematicians emerge from the washroom. The engineers are impressed.

When the conference has come to an end, the engineers decide that they are at least as smart as the mathematicians and also buy just one ticket for the whole group. This time, the mathematicians have no ticket at all. . . .

Again one of the mathematicians shouts: "Conductor coming!"

All of the engineers rush off to one washroom. One of the mathematicians goes to that washroom, knocks at the door, and says, "Your ticket, please. . . ."

Two math professors are sitting in a pub.

"Isn't it disgusting," the first one complains, "how little the general public knows about mathematics?"

"Well," his colleague replies, "you're perhaps a bit too pessimistic."

"I don't think so," the first one replies. "But anyhow, I have to go to the washroom now."

He leaves, and the other professor decides to use this opportunity to play a prank on his colleague. He makes a sign to the pretty, blonde waitress to come over.

"When my friend comes back, I'll wave you over to our table, and I'll ask you a question. I would like you to answer: x to the third over three. Can you do that?"

"Sure." The girl repeats several times: "x to the third over three, x to the third over three, x to the third over three. . . ."

When the first professor comes back from the washroom, his colleague says, "I still think you're way too pessimistic. I'm sure the waitress knows a lot more about mathematics than you imagine."

He makes her come over and asks her, "Can you tell us what the integral of x squared is?"

She replies: "x to the third over three."

The other professor's mouth drops wide open, and his colleague grins smugly when the waitress adds: ". . . plus C."

Back in the old days, when slide rules were still the most sophisticated computing equipment available to scientists and engineers. . .

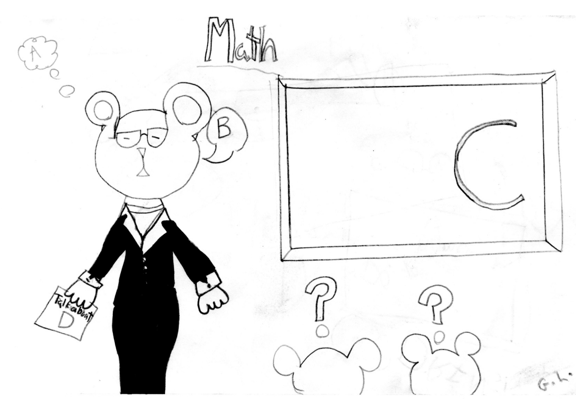
Engineering students are taking a math final. Of course, slide rules are not allowed. And, of course, someone is cheating and has brought a slide rule to the exam. He is hiding it under his desk, but the student sitting to his left—who is stuck on a difficult calculation—has noticed it.

"Hey," he whispers. "Can you help me? What's three times six?"

His classmate reaches for his slide rule, and after a few seconds replies, "19."

"Are you sure?"

The other student reaches again for his slide rule, and after another few seconds replies, "You're right. It's closer to 18—18.3, to be precise."



"He thinks A,
he says B,
he writes C,
and in fact he's talking about D."

Mathematician Laurent Schwartz,
describing his teacher at the Sorbonne.

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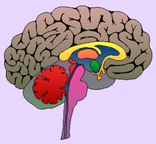
Sent by Gabrielle Lamoureux



Math Challenges

Problem 1. If a, b, c are the sides of a triangle, then

$$\frac{a}{b+c-a}+\frac{b}{c+a-b}+\frac{c}{a+b-c}\geq 3.$$

Problem 2. If ABC is an acute triangle with angles α, β and γ, and $p \geq 1$, then

(i) $\tan^p\alpha+\tan^p\beta+\tan^p\gamma \geq 3\sqrt{3^p}$.

(ii) $\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}\leq\frac{3\sqrt{3}}{8}$.

Problem 3. Let $a_1, a_2, \ldots, a_n > 0$ and $s = a_1+\ldots+a_n$. Show that

$$\frac{s}{s-a_1}+\frac{s}{s-a_2}+\ldots+\frac{s}{s-a_n}\geq\frac{n^2}{n-1}.$$

Problem 4. Let $a_1, a_2, \ldots, a_n > 0$, $m \geq 0$ and

$$A=\frac{a_1}{ma_1+a_2+\ldots+a_n}+\ldots+\frac{a_n}{a_1+a_2+\ldots+ma_n}.$$

Prove that

(i) If $m\in[0,1]$ then $A\geq\frac{n}{n+m-1}$;

(ii) If $m\in[1,\infty)$ then $A\leq\frac{n}{n+m-1}$.

Problem 5. If $a_k, x_k > 0$, $k = 1, 2, \ldots, n$, then

$$a_1x_1^{\frac{1}{a_1}}+\ldots+a_nx_n^{\frac{1}{a_n}}\geq(a_1+\ldots+a_n)(x_1\ldots x_n)^{\frac{1}{a_1+\ldots+a_n}}.$$

Problem 6. If $a, b > 0$, $p \geq 1$ and $x\in(0,\frac{\pi}{2})$, then

$$a(\sin x)^{\frac{1}{p}}+b(\cos x)^{\frac{1}{p}}\leq\left(a^{\frac{2p}{2p-1}}+b^{\frac{2p}{2p-1}}\right)^{\frac{2p-1}{2p}}.$$

Problem 7. If $a_1, a_2, \ldots, a_n\in\mathbb{R}$, then

$$\sin a_1\sin a_2\ldots\sin a_n+\cos a_1\cos a_2\ldots\cos a_n\leq 1.$$

Problem 8. Let $a_1, a_2, \ldots, a_n > 0$ be such that $a_1a_2\ldots a_n = 1$, and let $s = 1+a_1+a_2+\ldots a_n$. Prove that

$$\frac{1}{s-a_1}+\frac{1}{s-a_2}+\ldots+\frac{1}{s-a_n}\leq 1.$$

Problem 9. Let $f:[a,b]\to\mathbb{R}$ be a convex function. Then, for every x, y, $z\in[a,b]$, we have (*Popoviciu's Inequality*):

$$\begin{aligned}&\tfrac{1}{3}[f(x)+f(y)+f(z)]+f\left(\tfrac{x+y+z}{3}\right)\\ \geq\ &\tfrac{2}{3}\left[f\left(\tfrac{x+y}{2}\right)+f\left(\tfrac{y+z}{2}\right)+f\left(\tfrac{x+z}{2}\right)\right].\end{aligned}$$

Send your solutions to π *in the Sky*: Math Challenges.

Solutions to the Problems Published in the June, 2001 Issue of π *in the Sky*:

Problem 1. (By Edward T.H. Wang from Waterloo)
Note first that the inequality

$$\sqrt{2\sqrt{3\sqrt{4\ldots\sqrt{n}}}}<3$$

makes sense only if $n \geq 2$. Following the hint, we prove a more general inequality:

$$\sqrt{(a+2)\sqrt{(a+3)\sqrt{4\ldots\sqrt{(a+n)}}}}<a+3 \tag{1}$$

for all integers $n\geq 2$ and for all $a\in[0,\infty)$.
For $n = 2$, the inequality is clearly true. Suppose that (1) holds for $n = k$, $k \geq 2$ and for all $a\in[0,\infty)$. By replacing a with $a + 1$, we get

$$\sqrt{(a+3)\sqrt{(a+4)\ldots\sqrt{(a+1+k)}}}<a+4. \tag{2}$$

By multiplying both sides of (2) by $a+2$ and taking square roots of both sides, we get

$$\sqrt{(a+2)\sqrt{\ldots\sqrt{(a+1+k)}}}<\sqrt{(a+2)(a+4)}. \tag{3}$$

Since $\sqrt{(a+2)(a+4)}<a+3$, it follows from (3) that (1) holds for $n = k+1$ and $a\in[0,\infty)$.

Problem 2. (By Yuming Chen and Edward T.H. Wang from Waterloo)
The inequality

$$(a_1+a_2+\ldots+a_n)^2\leq a_1^3+a_2^3+\ldots+a_n^3 \tag{4}$$

holds only if all the integers $a_1, a_2, \ldots, a_n$ are distinct. For $n = 1$, we have $a_1^2\leq a_1^3$, which is clearly true since $a_1\geq 1$. Suppose that (4) holds for $n = k$; we will show that it also holds for $n = k+1$. Let $a_1, a_2, \ldots, a_{k+1}$ be distinct positive integers. We can assume that $1\leq a_1<a_2<\ldots<a_k<a_{k+1}$. Then, clearly $a_{k+1}\geq k+1$, and, by induction assumption, we have

$$\begin{aligned}\left[\sum_{l=1}^{k+1}a_l\right]^2&=(a_1+a_1+\ldots+a_k)^2+2a_{k+1}\sum_{l=1}^{k}a_l+a_{k+1}^2\\&\leq a_1^3+a_2^3+\ldots+a_k^3+2a_{k+1}\sum_{l=1}^{k}a_l+a_{k+1}^2.\end{aligned}$$

We need to show that $2a_{k+1}\sum_{l=1}^{k}a_l+a_{k+1}^2\leq a_{k+1}^3$, or

$$2\sum_{l=1}^{k}a_l+a_{k+1}\leq a_{k+1}^2. \tag{5}$$

To show (5), note that $a_n\leq a_{k+1}-1$, $a_{k-1}\leq a_{k+1}-2$, $\ldots$, $a_1\leq a_{k+1}-k$, hence

$$\sum_{l=1}^{k}a_l\leq ka_{k+1}-\sum_{l=1}^{k}l=ka_{k+1}-\frac{k(k+1)}{2},$$

so

$$2\sum_{l=1}^{k}a_l+a_{k+1}\leq(2k+1)a_{k+1}-k(k+1),$$

and therefore,

$$\begin{aligned}a_{k+1}^2-2\sum_{l=1}^{k}a_l-a_{k+1}&\geq a_{k+1}^2-(2k+1)a_{k+1}+k(k+1)\\&=(a_{k+1}-k)(a_{k+1}-(k+1))\geq 0.\end{aligned}$$

Consequently, we get (4). Notice that the equality holds if and only if $\{a_1, a_2, \ldots, a_n\} = \{1, 2, \ldots, n\}$. In this case, we obtain the well-known inequality

$$(1+2+\ldots+n)^2=1^3+2^3+\ldots+n^3.$$

Problem 3.
If $n = 1$, then the plane is divided into two parts and clearly the statement is true. Assume that it holds for $n = k$, and let's show that it stays true for $n = k+1$. Any k lines divide the plane into $1+\frac{k(k+1)}{2}$ regions. These k lines intersect the $(k+1)$-th line at k distinct points and none of these points coincides with the intersection points of the k lines. Therefore, $k+1$ new distinct regions of the plane are obtained. The total number of regions is

$$1 + \frac{k(k+1)}{2} + (k+1) = 1 + \frac{(k+1)(k+2)}{2},$$

and the statement follows from PMI (Principle of Mathematical Induction).

Problem 4.
For $n = 1$, the statement is clearly true. Assume that it is true for $n = k$ and let's prove it for $n = k+1$. The first k circles divided the plane into at most $k^2 - k + 2$ regions. The $(k+1)$-th circle intersects the k circles at most at $2k$ points (two different circles intersect at most at two points). These extra points divide the $(k+1)$-th circle into at most $2k$ parts and each of these parts divides the old region into two regions. Therefore, we obtain that there can be at most

$$(k^2 - k + 2) + 2k = (k+1)^2 - (k+1) + 2$$

regions. The statement follows by PMI.

Problem 5. For $n = 1$, the property is clear. Assume that the property is valid for $n = k$ and let's prove that it remains true for $n = k+1$. Any k circles divide the plane into parts that can be coloured properly. Now, consider the $(k+1)$-th circle and make the following recolouring: the parts outside the circle keep their initial colour and the parts inside it change their colours.

Problem 6. (By Edward T.H. Wang from Waterloo)
By putting $n = 2$ in (b) and using (a), we find that $\frac{1}{f(2)} = \frac{1}{2}$, so $f(2) = 2$. Similarly, we find $f(3) = 3$, $f(4) = 4$, etc. We now use PCI (Principle of Complete Induction—Math Strategies, June 2001 issue) to show that $f(n) = n$ for all $n \in \mathbb{Z}_+$. It remains to show that if $f(1) = 1$, $f(2) = 2$, $\ldots$, $f(k) = k$ for some $k \geq 2$, then $f(k+1) = k+1$. By using (b), we get

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{k\cdot f(k+1)} = \frac{k}{k+1}.$$

Since

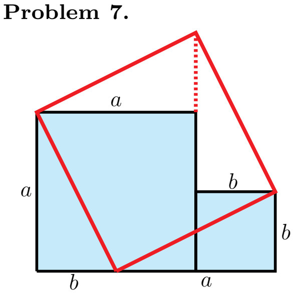
$$\begin{aligned}\tfrac{1}{1\cdot 2} + \tfrac{1}{2\cdot 3} + \ldots + \tfrac{1}{(k-1)k} &= (1-\tfrac{1}{2}) + (\tfrac{1}{2}-\tfrac{1}{3}) + \ldots + (\tfrac{1}{k-1}-\tfrac{1}{k})\\ &= 1 - \tfrac{1}{k} = \tfrac{k-1}{k}.\end{aligned}$$

Consequently, we get

$$\frac{1}{k\cdot f(k+1)} = \frac{k}{k+1} - \frac{k-1}{k} = \frac{1}{k(k+1)},$$

from which we obtain $f(k+1) = k+1$.

Problem 7.



For $n = 2$, there are several solutions.† You can see one solution in the diagram on the left. Assume that the statement is true for $n = k \leq 2$. We will show that it is also true for $n = k+1$. Consider $k+1$ squares, choose any two of them, and by dissection combine them into a bigger square (see the figure on the left). After this step, we are left with only k squares, and the inductive assumption applies.

† For example, see the web site at:
http://www.cut-the-knot.com/Generalization/cuttingsquare.html.

Math Links

Online Dictionary of Mathematics

ERIC WEISSTEIN'S world of MATHEMATICS

MathWorld Is Back! After more than a year's absence, Eric Weisstein's MathWorld, **the web's most extensive mathematical resource**, returns to the Internet on November 6, 2001.

http://mathworld.wolfram.com/

Ed Pegg's Puzzle Pages

This site celebrates math puzzles and mathematical recreations.

http://www.mathpuzzle.com/

Math Forum @ Drexel

Drexel University's enormous collection of educational math resources: key issues in math, math education, math resources by subject, math literature, discussion groups, etc.

http://mathforum.org/

Ivar Peterson's MathTrek

Ivar Peterson's weekly columns on a large variety of mathematical topics.

The Mathematical Association of America

http://www.maa.org/news/mathtrek.html

The Grey Labyrinth

THE GREY LABYRINTH

A collection of mostly mathematical puzzles, with new ones added about once a week.

http://www.greylabyrinth.com/index.htm

Our School Visit

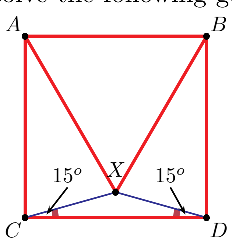
π *in the Sky* at Tempo School

by Don Stanley

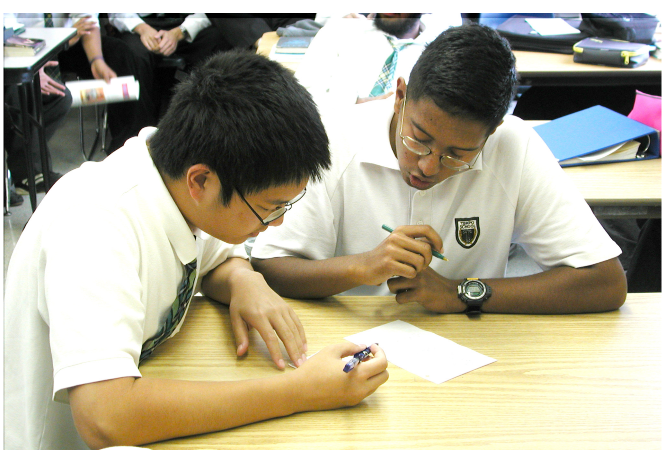
On October 23, 2001, five University of Alberta mathematicians invaded the grade 10 and 11 classes at Tempo School in Edmonton. They came to talk about mathematics and the π *in the Sky* magazine.

D. Hrimiuc, W. Krawcewicz, V. Runde and D. Stanley meeting students.

Dragos Hrimiuc offered a \$5 prize to anyone who could solve the following geometry problem:



Consider a square $ABCD$ with a point X inside it such that only the marked angles are known (see the diagram at the left). Show that triangle ABX is **equilateral**.



Students trying hard to solve the problem.

During the visit, some hard proofs were presented to show that math can be fun and interesting at the same time.

Solving problems can be really enjoyable.

Some students discover π *in the Sky*.

Dragos Hrimiuc smiles as he realizes that he will get to keep his \$5.

On behalf of the editorial board of π *in the Sky*, we would like to thank the principal of Tempo School, Dr. Kapoor, for making this visit possible.