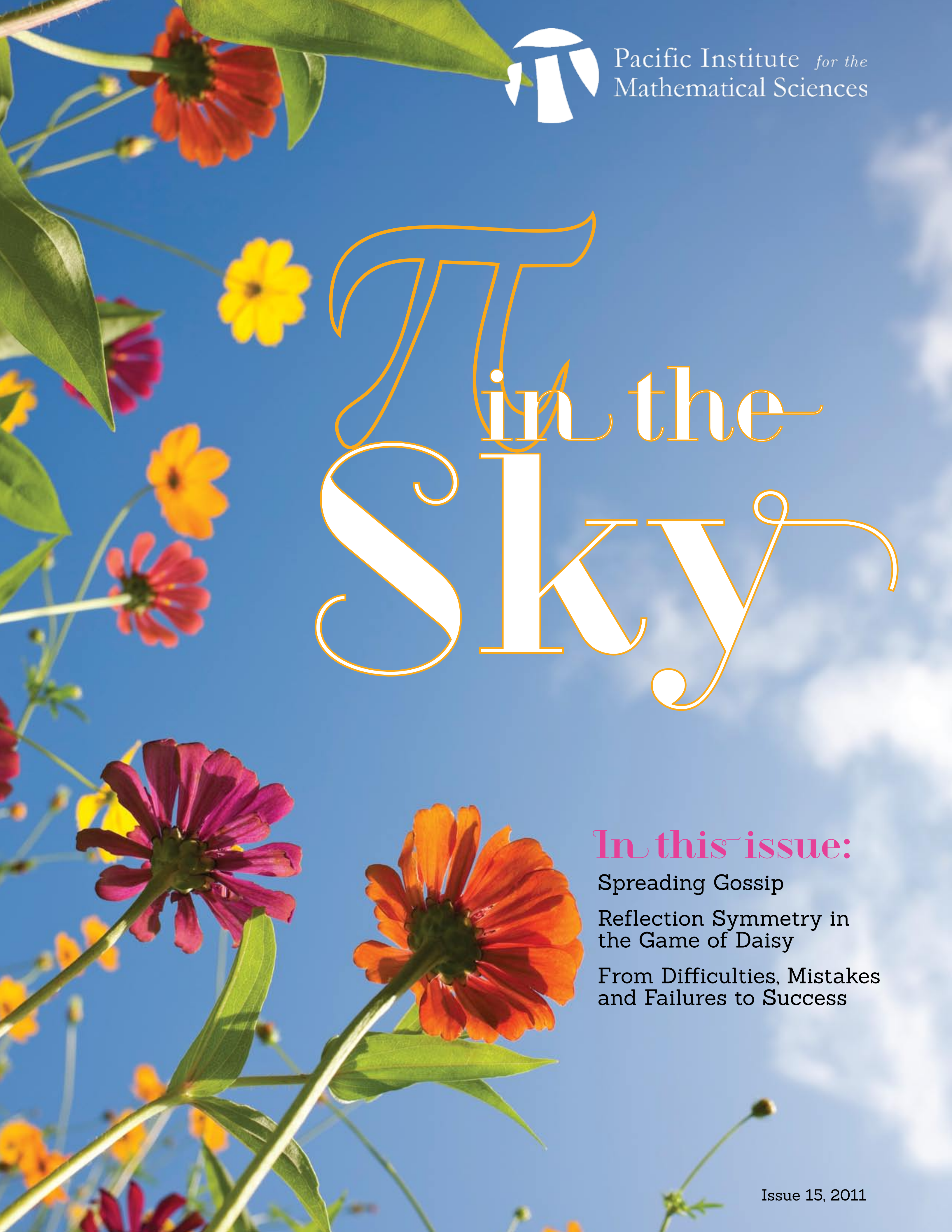

Pacific Institute *for the* Mathematical Sciences

π in the Sky

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From Difficulties, Mistakes and Failures to Success

Issue 15, 2011

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Spreading Gossip

Andy Liu, *PIMS Education Coordinator*, University of Alberta

Each of n friends has a juicy piece of gossip, and is eager to share with all the others. In each hour, pairs of friends engage in phone conversations. Each conversation involves exactly two friends, and each friend is involved in at most one conversation in each hour. In each conversation, the two participants tell each other every piece of gossip known so far. What is the minimum number $f(n)$ of hours for every friend to know every piece of gossip?

If there is only 1 person, no calls need to be made, so that $f(1) = 0$. If there are two friends, one call is both necessary and sufficient. Hence $f(2) = 1$. Suppose there are three friends A, B and C. In the first hour, two of them exchange gossip, say A and B. In the second hour, one of them, say B, calls C. Now both B and C know everything, but A does not. So we need a third hour during which one of B and C, say C, calls A. Hence $f(3) = 3$.

One might expect that $f(4)$ to be at least 3. However, we have $f(4) \leq 2$. Let the friends be A, B, C and D. In the first hour, A calls B and C calls D. In the second hour, A calls C and B calls D. Then everyone knows everything. Similarly, $f(8) \leq 3$. Let the additional friends be E, F, G and H. In the first two hours, they follow the moves of A, B, C and D. In the third hour, A calls E, B calls F, C calls G and D calls H. Again, everyone knows everything. These two upper bounds are actually exact values, and we can generalize to the following result.

Theorem 1. If $n = 2^k$ for some non-negative integer k, then $f(n) = k$.

Proof:
Each friend starts with 1 piece of gossip. In an hour, the number of pieces can double at most. Thus it takes at least k hours to learn all n pieces. The task may be accomplished in k hours. In the first hour, form pairs of two friends and have them call each other. In the second hour, form quartets of two pairs and have the two friends in one pair call the two friends in the other pair. In the third hour, form octets of two quartets and have the four friends in one quartet call the four friends in the other quartet. This process is continued until the k-th hour, when the friends in one half call the friends in the other half.

Consider now values of n lying between two powers of 2, say $2^k < n < 2^{k+1}$ for some non-negative integer k. The following examples show that $f(5) \leq 4$, $f(6) \leq 3$ and $f(7) \leq 4$. In each chart, we list the calls hour by hour, and the pieces of gossip known to each friend.

Hour	1	2	3	4
Calls	AB CD	BC DE	CD EA	AB
A	AB	AB	ABCDE	
B	AB	ABCD	ABCD	ABCDE
C	CD	ABCD	ABCDE	
D	CD	CDE	ABCDE	
E	E	CDE	ABCDE	

Hour	1	2	3
Calls	AB CD EF	BC DE FA	CD EF AB
A	AB	EFAB	ABCDEF
B	AB	ABCD	ABCDEF
C	CD	ABCD	ABCDEF
D	CD	CDEF	ABCDEF
E	EF	CDEF	ABCDEF
F	EF	EFAB	ABCDEF

Hour	1	2	3	4
Calls	AB CD EF	BC DE FG	CF DG EA	BF DE AG
A	AB	AB	ABCDEF	ABCDEFG
B	AB	ABCD	ABCD	ABCDEFG
C	CD	ABCD	ABCDEFG	
D	CD	CDEF	CDEFG	ABCDEFG
E	EF	CDEF	ABCDEF	ABCDEFG
F	EF	EFG	ABCDEFG	
G	G	EFG	CDEFG	ABCDEFG

As it turns out, these lower bounds are also exact values. They suggest that we should treat separately the case where n is odd and the case where n is even. Since these constructions are rather ad hoc, we give more general ones below.

Theorem 2. If $2^k < n < 2^{k+1}$ for some non-negative integer k, then $f(n) = k + 2$ if n is odd.

Proof:

The argument in Theorem 1 shows that in k hours, nobody knows every piece of gossip. In the $(k + 1)$-st hour, someone must sit out and still does not know everything. Hence $k + 2$ hours are necessary. To show that $k + 2$ hours are sufficient, let $m = n - 2^k$. Then $m < 2^k$. Choose m special friends. In the first hour, each special friend calls one of the other 2^k friends. These 2^k friends are in possession of all pieces of gossip. In the next k hours, using the construction in Theorem 1, every one of them will know everything. In the last hour, those who were called by special friends call the special friends back and tell them everything.

Theorem 3. If $2^k < n < 2^{k+1}$ for some non-negative integer k, then $f(n) = k + 1$ if n is even.

Proof:
The argument in Theorem 1 shows that in k hours, nobody knows every piece of gossip. Hence $k + 1$ hours are necessary. To show that $k + 1$ hours are sufficient, let $n = 2m$ so that $2^{k-1} < m < 2^k$. Label the friends $A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m$. For $1 \le i \le k$, let A_1 call $B_{2^{i-1}}$ in the i-th hour. In the $(k + 1)$-th hour, A_1 calls B_1 again. Whenever A_1 calls B_j, A_2 will call B_{j+1}, A_3 will call B_{j+2}, and so on, with the indices reduced modulo m if necessary. By symmetry, if A_1 knows all pieces of gossip, so will the other As. Since each A calls a different B in the last hour, all the Bs will know everything too. Thus we only need to focus on A_1, and by extension B_1.

Denote the pair of gossips known to A_j and B_j by P_j, $1 \le j \le m$. After talking to B_1, A_1 knows P_1. After talking to B_2, A_1 also knows P_2. After talking to B_4, A_1 will know P_3 and P_4. After talking to B_8 A_1 will also know P_5, P_6, P_7 and P_8. By the time A_1 has talked to $B_{2^{k-1}}$, A_1 will know all of P_1 to $P_{2^{k-1}}$. Meanwhile, after talking to A_1, B_1 will be talking to $A_m, A_{m-2}, A_{m-6}, \ldots, A_{m-2^{k-1}+2}$, and will learn gossips $P_m, P_{m-1}, P_{m-2}, \ldots, P_{m-2^{k-1}+2}$. Note that $m - 2^{k-1} + 2 \le 2^{k-1} + 1$ since $m < 2^k$. Hence A_1 and B_1 will know everything between them when they talk for the second time.

We give an illustration with the case $n = 14$. The chart below shows the calls made during each hour. By the third hour, A_1 knows P_1 from B_1, P_2 from B_2, and P_3 and P_4 from B_4, while B_1 knows P_1 from A_1, P_7 from A_7, and P_5 and P_6 from A_5.

Friends	A_1	A_2	A_3	A_4	A_5	A_6	A_7	B_1	B_2	B_3	B_4	B_5	B_6	B_7
First Hour	B_1	B_2	B_3	B_4	B_5	B_6	B_7	A_1	A_2	A_3	A_4	A_5	A_6	A_7
Second Hour	B_2	B_3	B_4	B_5	B_6	B_7	B_1	A_7	A_1	A_2	A_3	A_4	A_5	A_6
Third Hour	B_4	B_5	B_6	B_7	B_1	B_2	B_3	A_5	A_6	A_7	A_1	A_2	A_3	A_4
Fourth Hour	B_1	B_2	B_3	B_4	B_5	B_6	B_7	A_1	A_2	A_3	A_4	A_5	A_6	A_7

For special cases of our result, see the references below.

[1] Dennis Shasha, The Puzzling Adventures of Dr. Ecco, Dover Publications Inc., Mineola NY, 1998, pp. 62–64 and 156.

[2] Peter Taylor, International Mathematics Tournament of the Towns 1980–1984, Australian Mathematics Trust, Canberra, 1993, pp. 31–32, 37 and 40–41.

Reflection Symmetry in the Game of Daisy

By Karen Wang, Lynbrook High School

Symmetry has interesting applications in the world of mathematics—more specifically, the area of mathematics involving game analysis. Let's explore an example of reflection symmetry in the form of the Daisy game.

This game requires two players and a thirteen-petal flower (Fig. 1). The rules are as follows: each player can choose to remove either one or two adjacent petals on his or her turn. Players alternate turns, and the person to remove the last petal wins. Seems simple enough, but the question is: what if you could guarantee a win every time?

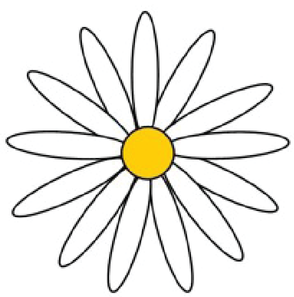


Fig. 1

Let's call the two players Alice and Bob; Alice gets the first move. If the rules allowed players to pick only one petal at a time, this game would be too easy! Since there is an odd number of petals, Alice would always win. However each player can choose either one or two, thus complicating matters. The truth is that in the game of Daisy, if Bob, the second player, knows what he is doing, Alice cannot win.

We'll examine the game from Bob's point of view. Alice makes the first move—we'll assume that she removed one petal (Fig. 2).

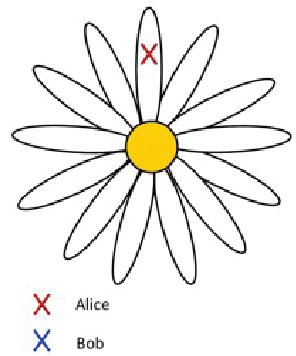


Fig. 2

Now Bob is looking at the remaining twelve petals, thinking, I have a chance to turn the tables in my favor! But which petals, and how many, should I pick? Well, if Bob picks one petal at a time, Alice will win no matter what. Thus, Bob wants to choose petals in such a way as to leave an even amount remaining. Easier said than done; since he can only pick two petals that are adjacent to each other, he might end up in a situation where there is an even number of petals on his turn, but all of them are not adjacent to any other petal. Consider Figure 3 below, which we will observe as an artbitrary but possible outcome of playing out the game. In this situation, Bob cannot pick two petals at once, since the rules dictate that the two petals must be adjacent.

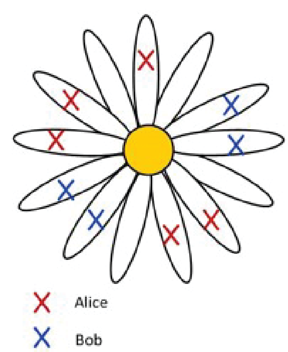


Fig. 3

In this case Bob would have no choice but to create an odd remainder of petals, beginning his spiral into doom. But he shouldn't despair! The truth is, as the second player, Bob has an advantage— he can control the consequences of Alice, the first player's actions. No matter what Alice does, Bob wants to be able to create an even number of petals on his turn and eventually force Alice to create an odd remainder. This is where the interesting properties of reflection symmetry come into play. Let me demonstrate.

Let's say Alice chooses one petal on her first move. Then let Bob choose two petals—the two petals directly opposite of Alice's (Fig. 4).

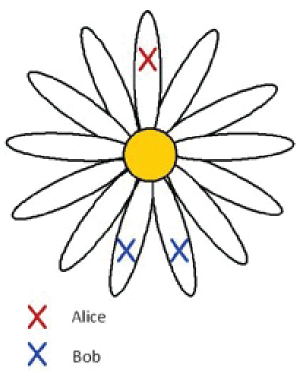


Fig. 4

By doing this, Bob has reconfigured the structure of the game. The daisy is now divided into two equal sections, with five available petals remaining in each section (Fig. 5).

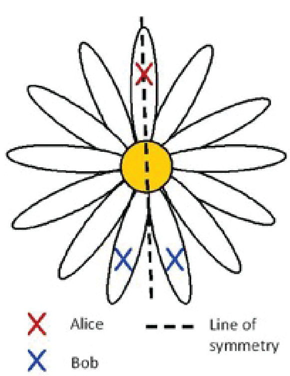


Fig. 5

Now that the game has a symmetrical configuration, what should Bob's next move be? If you think about it, a system in reflection symmetry allows him to do one thing: maintain the symmetry by mirroring any disturbances that are caused to the balance. These disturbances are, of course, the moves of the first player, Alice. Whatever Alice does, whether she chooses one or two petals, and whether she chooses them on the left or right side Bob can maintain the symmetry by mirroring her movements. That is, Bob will now choose the same number of petals directly opposite Alice's previous move. For example, Figure 6 below illustrates Alice's possible arbitrary selection of two petals, and Bob's subsequent mirroring of her movement.

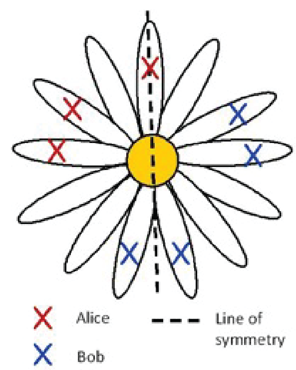


Fig. 6

Eventually, Alice will reach her turn and face a situation in which there are only two non-adjacent petals remaining (Fig. 7), or two non-adjacent pairs of petals, one pair on either side.

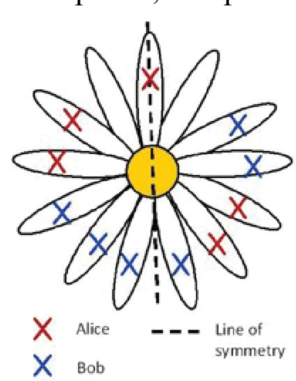


Fig. 7

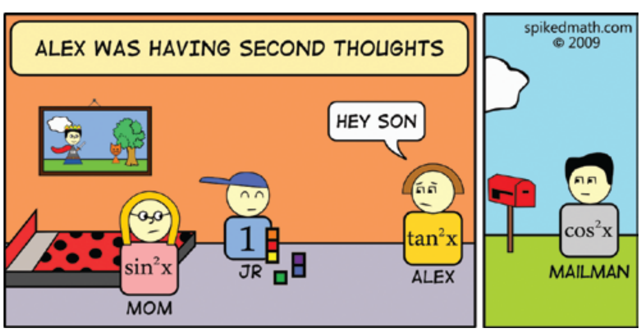
Since she can only remove one petal at a time in this case, Bob is guaranteed to remove the last petal.

Of course this is all based on the assumption that Alice, the first player, removes one petal first. But even if Alice removes two, the second player's win is still guaranteed. He just needs to create symmetry by removing the single petal opposite the ones Alice has chosen. The main goal is to create a flower with reflection symmetry; the order of petals removed at first doesn't matter. The rest is the same process.

So why does reflection symmetry work? Basically, the second player, Bob, wants to divide the daisy into two equal "sides", and he aims to force Alice to empty one of the sides first.

After Bob establishes the symmetry, Alice will be able to take petals from either the right or left side of the daisy. But by mimicking Alice's moves, Bob can still maintain the concept of two equal "sides". He maintains the balance, so to speak, and force Alice to disrupt the symmetry. The remainder will always be even, so eventually Alice will have no choice but to empty one of the sides. When Bob makes the symmetric move, he will automatically take the last petal and win the game.

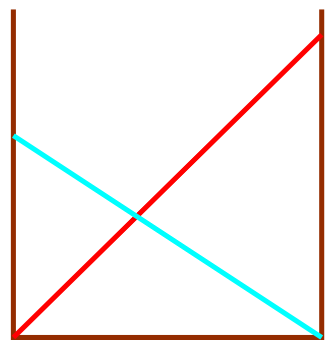
As a final thought, try to find a winning strategy for a 14-petal daisy!



GEOMETRIC PROBLEMS

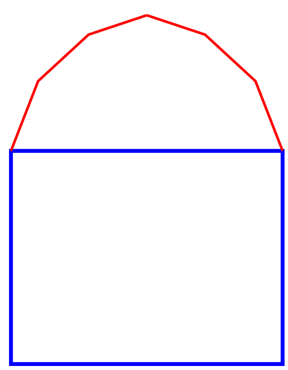
Here are two geometric problems submitted by readers.

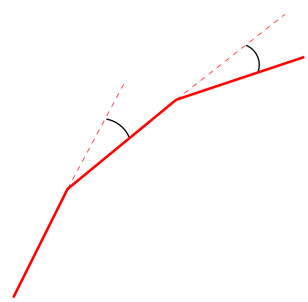
This one is based on a suggestion of **Terence Coates.**



In a narrow alley a 35 foot ladder is placed up against one wall with its base touching the opposite wall. A 29 foot ladder is placed against the second wall with its base touching the first wall. The two ladders cross 11 2/3 feet above the ground (see the Figure). How wide is the alley?

And this one was suggested by **Gregory Akulov.**





The roof of a house is in flat parts with three different slopes: 7, *m* and 1 (the first three from left to right). If the the angle between each pair of slopes is the same (as shown in the figure), what is *m*?

FROM ~~DIFFICULTIES, MISTAKES, AND FAILURES~~ TO *Success*

EXAMPLES OF ADJUSTMENTS IN THE PROCESS OF MATHEMATICAL PROBLEM SOLVING

By Luo Qi*

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ABSTRACT Based on the analysis of three cases of mathematical problem solving, this paper shows how we should deal with the circumstances arising during the process of problem solving in order to succeed when we face difficulties, mistakes and failures.

KEY WORDS Difficulty; mistake; failure; success; problem solving; analysis

In our problem solving experiences, we actually encounter difficulties, mistakes, and failures much more often than the excitement of success. However, difficulties, mistakes and failures should not only leave us with negative results. When we face difficulties, we should adjust the problem-solving orientation to resolve difficulties or challenge ourselves to overcome the difficulties. When we face mistakes, we must know how to discover and correct them promptly; when we face failure, it is necessary to find the reasons and turn failure into success. Here, we will demonstrate how to succeed from difficulties, mistakes and failures by illustrating the process of analyzing three common mathematical problem solving techniques.

Case 1: Resolve and overcome ~~DIFFICULTIES~~

Example 1. Find the value of $\dfrac{\cos 20^\circ}{2\sin 80^\circ - \sin 20^\circ}$

Attempted solution from students:

$$\frac{\cos 20^\circ}{2\sin 80^\circ - \sin 20^\circ} = \frac{\cos 20^\circ}{2\cos 10^\circ - \sin 20^\circ} = \frac{\cos 20^\circ}{2\cos 10^\circ - 2\sin 10^\circ \cos 10^\circ} = \frac{\cos 20^\circ}{2\cos 10^\circ(1-\sin 10^\circ)}$$

It seems difficult to further simplify this term. Facing difficulties, we may adjust the direction of problem solving to resolve difficulties, and then observe the pattern of the numbers in the expression again. Observing that

$\sin 80^\circ = \sin(60^\circ + 20^\circ) = \sin 60^\circ \cos 20^\circ + \cos 60^\circ \sin 20^\circ$, we have the following solution:

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$$\frac{\cos 20^\circ}{2\sin 80^\circ - \sin 20^\circ} = \frac{\cos 20^\circ}{2\sin(60^\circ + 20^\circ) - \sin 20^\circ} = \frac{\cos 20^\circ}{2\sin 60^\circ \cos 20^\circ + 2\cos 60^\circ \sin 20^\circ - \sin 20^\circ} = \frac{\sqrt{3}}{3}$$

If we had manipulated the original expression into $\dfrac{\cos 20^\circ}{2\cos 10^\circ - \sin 20^\circ}$ then observed that

$\cos 10^\circ = \cos(30^\circ - 20^\circ) = \cos 30^\circ \cos 20^\circ + \sin 30^\circ \sin 20^\circ$ we could similarly find the answer quickly. Here faced with a difficult situation, we solve the problem easily by adjusting our problem-solving direction. As we say: Take a step back in order to move forward!

However, the understanding of a problem is different for different people, so that multiple solutions can be presented. Did the first exploration really not work? In fact, as soon as we realize we have done nothing wrong, we should be able to obtain correct results. All roads lead to Rome! Let us continue the original method for solving:

We change $1 - \sin 10^\circ$ to trigonometric product form and have

$$\begin{aligned}
&\frac{\cos 20^\circ}{2\sin 80^\circ - \sin 20^\circ} = \frac{\cos 20^\circ}{2\cos 10^\circ(1 - \sin 10^\circ)} = \frac{\cos 20^\circ}{2\cos 10^\circ(1 - 2\sin 5^\circ \cos 5^\circ)} \\
&= \frac{\cos 20^\circ}{2\cos 10^\circ(\sin 5^\circ - \cos 5^\circ)^2} = \frac{\cos 20^\circ}{2\cos 10^\circ(\sin 5^\circ - \sin 85^\circ)^2} = \frac{\cos 20^\circ}{2\cos 10^\circ(2\cos 45^\circ \sin 40^\circ)^2} \\
&= \frac{\cos 20^\circ}{4\cos 10^\circ \sin^2 40^\circ}.
\end{aligned}$$

Noting that $40^\circ = 2 \times 20^\circ$ and $\sin 40^\circ = 2\sin 20^\circ \cos 20^\circ$, we multiplied the expression by $\dfrac{2\sin 20^\circ}{2\sin 20^\circ}$ and get

$$\begin{aligned}
\frac{2\sin 20^\circ \cdot \cos 20^\circ}{2\sin 20^\circ \cdot 4\cos 10^\circ \sin^2 40^\circ} &= \frac{1}{8\sin 20^\circ \cos 10^\circ \sin 40^\circ} \\
&= \frac{1}{4(\sin 30^\circ + \sin 10^\circ)\sin 40^\circ} = \frac{1}{2\sin 40^\circ + 4\sin 10^\circ \cdot \sin 40^\circ} \\
&= \frac{1}{2\sin 40^\circ - 2(\cos 50^\circ - \cos 30^\circ)} = \frac{\sqrt{3}}{3}
\end{aligned}$$

We observe that although the process is more complicated, we can still find the answer! Moreover, we need to rely on our courage and daring to deal with difficulties and we do not give up easily when facing them. As the saying goes: take a further step and you will see unlimited scenery!

Case 2: Discover and correct ~~MISTAKES~~

Example 2. If S_n, S_n' are the sums of the first n items of the arithmetic sequences $\{a_n\}$ and $\{a_n'\}$ respectively and $S_n : S_n' = (2n+3):(3n+2)$ for all $n \geq 1$ then find $a_n : a_n'$.

Solution from students: First $S_n : S_n' = (2n+3):(3n+2)$, so we suppose

$$S_n = k(2n+3), \text{ and } S_n' = k(3n+2).$$

$$\text{Therefore, } a_n = S_n - S_{n-1} = k(2n+3) - k[2(n-1)+3] = 2k$$

$$a_n' = S_n' - S_{n-1}' = k(3n+2) - k[3(n-1)+2] = 3k.$$

Hence, the final solution is $a_n{:}a_n' = 2k{:}3k = 2{:}3$.

The above process of problem solving appears to be faultless. Reviewing the problem solving process we can find no use of the conditions of arithmetic sequences. Is the condition of an arithmetic sequence redundant? And $a_n{:}a_n' = 2{:}3$ has nothing to do with n! Therefore we use one particular value to test this:

Let $n = 1$ therefore $a_1{:}a_1' = 2{:}3 = S_1{:}S_1'$, but $S_1{:}S_1' = (2\times 1+3){:}(3\times 1+2) = 1{:}1$.

There is a conclusion which conflicts with the given condition! Where does the mistake appear? We carefully reflect on each step of the problem solving process and find nothing wrong with the operations. Therefore, the error could only lie in the assumption:

Suppose $S_n = k(2n+3)$, and $Sn' = k(3n+2)$

Recall the formula for the sum of the first n terms of an arithmetic sequence. Since the correct formula for S_n is $S_n = na_1 + \frac{n(n-1)}{2}d$, we can see that S_n is not a linear function of n but a quadratic function of n . Here, the fundamental mistake is to suppose that S_n is a linear function of n.

According to the problem conditions, it can be assumed that:

$S_n = (kn+b)(2n+3)$, $S_n' = (kn+b)(3n+2)$ (where k and b are constants); therefore,

$$a_n = S_n - S_{n-1} = (kn+b)(2n+3) - [k(n-1)+b][2(n-1)+3] = k(4n+1)+2b$$

$$a_n' = S_n' - S_{n-1}' = (kn+b)(3n+2) - [k(n-1)+b][3(n-1)+2] = k(6n-1)+3b$$

We have $a_n{:}a_n' = \dfrac{k(4n+1)+2b}{k(6n-1)+3b}$.

How can we eliminate the parameters k and b in the expression? What other conditions can be used?

We return to the formula for S_n again, $S_n = na_1 + \frac{n(n-1)}{2}d$.

With careful observation of the formula it can be found that S_n is not only a quadratic function, but also a function passing through (0,0). Hence, we substitute (0,0) into the expression, $S_n = (kn+b)(2n+3)$ and get $b = 0$.

The solution is: $a_n{:}a_n' = \dfrac{k(4n+1)}{k(6n-1)} = \dfrac{(4n+1)}{(6n-1)}$

Here the incorrect assumption leads us to assume $S_n = k(2n+3), S_n' = k(3n+2)$, neglecting the original condition. Copying and applying methods mechanically without carefully analyzing the specific conditions results in mistakes! By reflection we find the mistake, correct the errors in essence, make up for glitches and consolidate our basic knowledge.

If we consider the general term formula of an arithmetic sequence $a_n = a_1 + (n-1)d$ we have

$$\frac{a_n}{a_n'} = \frac{a_1 + (n-1)d}{a_1' + (n-1)d'} \tag{1}$$

For finding $a_n : a_n'$ what we need simply is to determine a_1, a_1', d and d' with the same letters.

We substitute $S_n = na_1 + \frac{n(n-1)}{2}d$ into the given expression $S_n : S_n' = (2n+3):(3n+2)$ directly and get:

$$\frac{S_n}{S_n'} = \frac{na_1 + \frac{n(n-1)}{2}d}{na_1' + \frac{n(n-1)}{2}d'} = \frac{2a_1 + (n-1)d}{2a_1' + (n-1)d'} = \frac{2n+3}{3n+2} \tag{2}$$

What should we do next? We are looking for the relationship of a_1, a_1', d and d.

We noted that (2) is valid for all the natural numbers, therefore, we can consider specific cases to deal with this. Let $n = 1,2,3$ so that $\frac{a_1}{a_1'} = \frac{1}{1}$ $\frac{2a_1 + d}{2a_1' + d'} = \frac{7}{8}$ $\frac{a_1 + d}{a_1' + d'} = \frac{9}{11}$. Solving this set of equations,

we get $a_1 = a_1' = \frac{5}{4}d, d' = \frac{3}{2}d$ (3)

Substituting (3) into (1) we have $\frac{a_n}{a_n'} = \frac{\frac{5}{4}d + (n-1)d}{\frac{5}{4}d + (n-1)\cdot\frac{3d}{2}} = \frac{4n+1}{6n-1}$.

Here, in order to avoid discussing S_n and S_n', we used the formula for the sum of the first n terms and the general term of the arithmetic sequence directly. This method is also suitable when we change from an arithmetic sequence to a geometric sequence!

Case 3: From ~~FAILURE~~ to success

Example 3. Find the minimum of the function $y = \frac{x^2 + a + 1}{\sqrt{x^2 + a}} (a > 1)$.

Solution from students: Transforming the function, we get

$y = \frac{1}{\sqrt{x^2 + a}} + \sqrt{x^2 + a} \geq 2\frac{1}{\sqrt{x^2 + a}} \cdot \sqrt{x^2 + a} = 2$. The minimum is 2.

But the equality holds only when $\frac{1}{\sqrt{x^2+a}} = \sqrt{x^2+a} \Rightarrow x^2 = 1-a$. This is contrary with the known condition $a > 1$. Hence, the minimum of the function is not less than 2, but we cannot deduce from this that the minimum actually is 2.

Now that we understand the reasons for failure, we will try to satisfy the condition by adjusting the coefficient of inequality to attain equality. How do we adjust it? We use the method of undetermined parameters to solve it.

Suppose $y = \frac{1}{\sqrt{x^2+a}} + \sqrt{x^2+a} = \frac{1}{\sqrt{x^2+a}} + \lambda\sqrt{x^2+a} + (1-\lambda)\sqrt{x^2+a}, 0 < \lambda < 1$

Therefore, using the arithmetic-geometric mean inequality, we get

$$y \geq 2\sqrt{\lambda} + (1-\lambda)\sqrt{x^2+a} \geq 2\sqrt{\lambda} + (1-\lambda)\sqrt{a}.$$

Here, the conditions to make the inequalities achieve equality at the same time are:

$$\begin{cases} \frac{1}{\sqrt{x^2+a}} = \lambda\sqrt{x^2+a} \\ x^2 = 0 \end{cases} \Rightarrow \begin{cases} \lambda = \frac{1}{a} \\ x = 0 \end{cases}$$

Therefore, the minimum of the function is: $2\sqrt{\frac{1}{a}} + (1-\frac{1}{a})\sqrt{a}$. Simplifying we get $\frac{(a+1)\sqrt{a}}{a}$.

In the same way if we suppose

$$y = \sqrt{x^2+a} + \frac{1}{\sqrt{x^2+a}} = \sqrt{x^2+a} + \frac{\lambda'}{\sqrt{x^2+a}} - \frac{\lambda'-1}{\sqrt{x^2+a}}, (\lambda' > 1)$$

Then we have $y \geq 2\sqrt{\lambda'} - \frac{\lambda'-1}{\sqrt{x^2+a}} \geq 2\sqrt{\lambda'} - \frac{\lambda'-1}{\sqrt{a}}$, so that the conditions to make

inequalities achieve equality at the same time are $\begin{cases} \sqrt{x^2+a} = \frac{\lambda'}{\sqrt{x^2+a}} \\ x^2 = 0 \end{cases} \Rightarrow \begin{cases} \lambda' = a \\ x = 0 \end{cases}$.

We can now see that minimum $\frac{(a+1)\sqrt{a}}{a}$.

Of course, we may also find other ways to solve the problem after the failure of the direct use of the basic inequality. The use of the monotonicity of a function is a common method of solving extreme value problems. Then, how do we tell whether the given function a monotonic function? We use derivatives to judge.

We find the derivative of the known function, $y' = \frac{x(x^2+a-1)}{(x^2+a)\sqrt{x^2+a}}$.

Taking note that $a > 1$ so that $y' \geq 0$ when $x \geq 0$. The function is a continuous monotone decreasing function when $x \geq 0$.

Similarly $y' < 0$ when $x < 0$. Of course, we can also use the definition of monotonicity to prove it.

Therefore the minimum of the function is the function value when $x = 0$.

The solution is $y_{\min} = \dfrac{0^2 + a + 1}{\sqrt{0^2 + a}} = \dfrac{(a+1)\sqrt{a}}{a}$.

The method of completing the square is also an important method for solving extreme value problems of a function. Can we solve the problem with this method?

Notice $\sqrt{x^2 + a} = \sqrt[4]{(x^2 + a)^2}$

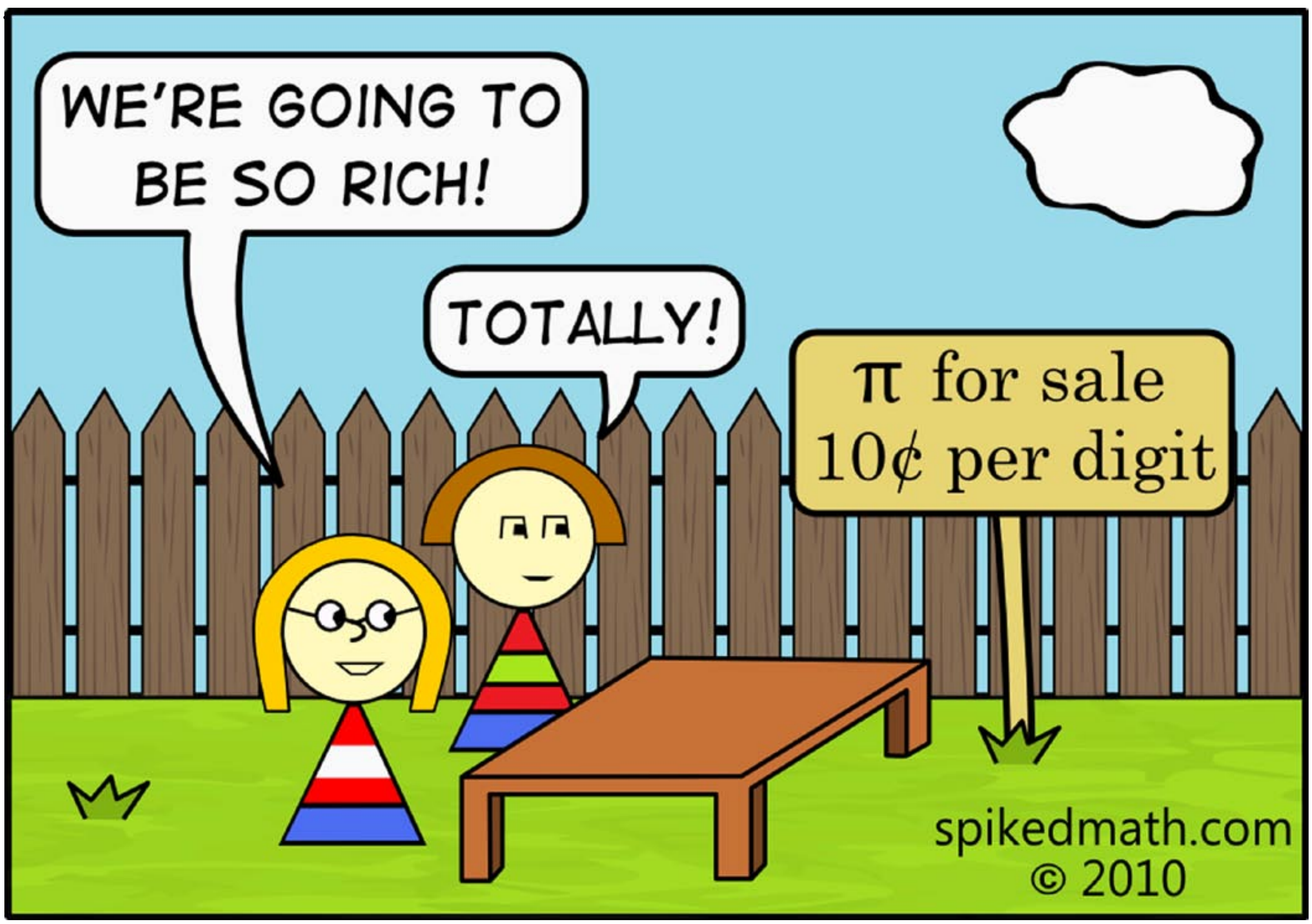
We have $y = \sqrt{x^2 + a} + \dfrac{1}{\sqrt{x^2 + a}} = (\sqrt[4]{x^2 + a})^2 + \dfrac{1}{(\sqrt[4]{x^2 + a})^2} - 2 + 2.$

As $\sqrt[4]{x^2 + a} - \dfrac{1}{\sqrt[4]{x^2 + a}} \geq \sqrt[4]{a} - \dfrac{1}{\sqrt[4]{a}} > 0$

So that $y = (\sqrt[4]{x^2 + a} - \dfrac{1}{\sqrt[4]{x^2 + a}})^2 + 2 \geq (\sqrt[4]{a} - \dfrac{1}{\sqrt[4]{a}})^2 + 2 = \dfrac{(a+1)\sqrt{a}}{a}$

Thus, the minimum is $\dfrac{(a+1)\sqrt{a}}{a}$.

It is therefore clear that, although failure is inevitable, it does not mean what we have done is worth nothing. There is maybe just one little step between failure and success. The next step might be successful so long as we strive on with confidence. In many cases, truth comes from error; success from failure. Even when we have really failed, we can still learn a good lesson from those failures and make them the mother of success!



Analytical Geometric Characterizations of Parabolas

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The main objective of this article (The Theorem) is to present two closely related analytical geometric characterizations of parabolas, as represented by quadratic functions. The first characterization simply asserts that for any two points $A\,(a, f(a))$ and $B\,(b, f(b))$ on the graph of a parabola, the slope of the secant segment AB is always the average between the slopes of the tangent lines at A and B to the curves. The second characterization is that the two such described tangent lines always meet (horizontally) right midway between A and B. These two closely related properties, when written in the form of algebraic relations between a and b, turns out to generalize to an interesting assertion (The Proposition) regarding how slopes of two intersecting tangent lines to the graph of a general differentiable function are related to the abscissa of the point of their intersection.

THEOREM: Let $f(x)$ be a ***nonlinear*** differentiable function over the real number line. Then the following conditions are equivalent:

(I) $f(x)$ is a quadratic function; that is $f(x)$ represents a parabola.

(II) For any two points $A\,(a, f(a))$ and $B\,(b, f(b))$ on the graph of $f(x)$, the slope of the secant segment AB is the average between the slopes of the two tangent lines to the graph of $f(x)$ at A and B. In algebraic form this means for any two points $A\,(a, f(a))$ and $B\,(b, f(b))$ on graph of $f(x)$ the following relation holds:

$$[f'(b) + f'(a)]/2 = [f(b) - f(a)]/(b - a). \tag{1}$$

(III) For any distinct points $A\,(a, f(a))$ and $B\,(b, f(b))$ on the graph of $f(x)$, the x-coordinate of the point of intersection of the two tangent lines to the graph of $f(x)$ at points $(a, f(a))$ and $(b, f(b))$ is exactly the average $(a + b)/2$ of the two x-coordinates of the points $(a, f(a))$ and $(b, f(b))$.

PROOF: (I) ⇨ (II) Verification of this part is only a routine algebraic manipulation. Let the points $A\,(a, f(a))$ and $B\,(b, f(b))$ be on the graph of a quadratic function $f(x) = c\,(x - h)^2 + k$. Then $f'(x) = 2c\,(x - h)$ and we have,

$$f'(a) = 2c\,(a - h)$$
$$f'(b) = 2c\,(b - h).$$

Therefore

$$[f'(b)+f'(a)]/2=[2c(a-h)+2c(b-h)]/2=c(a+b-2h)$$

$$\begin{aligned}[f(b)-f(a)]/(b-a)&=[c(b-h)^2+k-c(a-h)^2-k]/(b-a)\\&=c[(b-h)^2-(a-h)^2/(b-a)\\&=c[b^2-a^2)-2h(b-a)]/(b-a)\\&=c(b+a-2h)\end{aligned}$$

which shows the two sides of (1) are identical, and thus (II) follows.

(II) ⇨ (III) Let $f(x)$ be a nonlinear differentiable function satisfying the assertion (II). Then, given any two points $A\,(a, f(a))$ and $B\,(b, f(b))$ on the graph of $f(x)$, the respective equations of the tangent lines at these two points are as follows,

$$\begin{aligned}y&=f'(a)(x-a)+f(a)\\y&=f'(b)(x-b)+f(b)\end{aligned}$$

In order to find the x-coordinate of the point of intersection of the two tangent lines we need to solve the above system of linear equations for x. A simple calculation shows that

$$x=\frac{bf'(b)-af'(a)-[f(b)-f(a)]}{f'(b)-f'(a)}. \tag{2}$$

Since by assumption of part (II), (1) holds, and since we can first rewrite (1) in the form

$$[f(b)-f(a)]=(b-a)\frac{[f'(a)+f'(b)]}{2},$$

upon substitution of the above right hand side for $[f(b)-f(a)]$ in the numerator of the fraction in (2) we get

$$x=\frac{bf'(b)-af'(a)-(b-a)[f'(a)+f'(b)]/2}{f'(b)-f'(a)}$$

$$=\frac{2bf'(b)-2af'(a)-(b-a)[f'(a)+f'(b)]}{2[f'(b)-f'(a)]}=\frac{bf'(b)-af'(a)-bf'(a)+af'(b)]}{2[f'(b)-f'(a)]}$$

$$=\frac{[f'(b)-f'(a)](a+b)}{2[f'(b)-f'(a)]}=\frac{(a+b)}{2},$$

which means (III) follows.

(III) ⇨ (I) To this end, assume $f(x)$ is a differentiable function satisfying assertion (III).

Then, again, for two arbitrary points $A\ (a, f(a))$ and $B\ (b, f(b))$ on the graph of $f(x)$, equations of the tangent lines at these two points are,

$$y = f'(a)(x-a) + f(a)$$
$$y = f'(b)(x-b) + f(b).$$

Since by assumption of part (III) the x-coordinate of the point of intersection of the above two lines is $x = (a+b)/2$, when substituting this abscissa in the above two equations, the two right hand sides should be identical. That is,

$$f'(a)\left(\frac{a+b}{2} - a\right) + f(a) = f'(b)\left(\frac{a+b}{2} - b\right) + f(b),$$

or

$$f'(a)\left(\frac{b-a}{2}\right) + f(a) = f'(b)\left(\frac{a-b}{2}\right) + f(b).$$

Or

$$f'(a)(b-a) + 2f(a) = f'(b)(a-b) + 2f(b). \tag{3}$$

Since (3) should hold for any two real numbers a and b, we can keep the parameter a fixed, say $a = 0$ and let $b = x$ vary over all ***nonzero*** real numbers. Then (3) becomes

$$xf'(0) + 2f(0) = -xf'(x) + 2f(x).$$

Using the usual notations $y = f(x)$ and $y' = f(x)$ we obtain a linear differential equation as follows

$$xy' - 2y = -xf'(0) - 2f(0).$$

We now divide both sides of this last equation by x^3, and write the equation as follows

$$\frac{x^2y' - 2xy}{x^4} = \frac{-f'(0)}{x^2} - \frac{2f(0)}{x^3}.$$

Or

$$\frac{d}{dx}\left(\frac{y}{x^2}\right) = \frac{-f'(0)}{x^2} - \frac{2f(0)}{x^3},$$

which (upon integration of both sides) implies

$$\frac{y}{x^2} = \frac{f'(0)}{x} + \frac{f(0)}{x^2} + C.$$

Multiplying both sides by x^2 implies that $y = f(x)$ must be the quadratic function

$$f(x) = f'(0)x - f(0) + Cx^2$$

This completes the proof of the theorem.

Note that in the above proof C can not be zero, otherwise the function will be linear, contrary to the assumption of the theorem.

REMARK Equivalence of parts (II) and (III) in the above theorem can be generalized into the following interesting skew-type property between two tangent lines to the graph of any function, ***when they intersect***. Here I take the point of view that the number $(a + b)/2$ in part (III) and the number $[f'(a) + f'(b)]/2$ in part (II) are only one of the zillions ***corresponding*** linear combinations of the respective numbers a and b in part (III) and of numbers $f'(a)$ and $f'(b)$ in part (II).

PROPOSITION Let $f(x)$ be a differentiable function and $A(a, f(a))$, $B(b, f(b))$ be arbitrary points on its graph, with their respective intersecting tangent lines T_a and T_b. If c denotes the abscissa of the point of intersection of T_a and T_b, then for any given real number s, the relation $c = sa + (1 - s)b$ holds if and only if

$$\frac{f(b) - f(a)}{b - a} = (1 - s)f'(a) + sf'(b).$$

This means the same linear combination expressing the slope of the secant segment AB in terms of the respective slopes $f'(a)$ and $f'(b)$ of T_a and T_b will determine how the abscissa of the point of intersection of the lines T_a and T_b can be expressed in terms of a and b (in a skew manner as seen above).

PROOF: Since (as we saw in the proof (II)⇨(III) of the Theorem) the abscissa of the point of intersection of T_a and T_b is

$$c = \frac{bf'(b) - af'(a) - f(b) + f(a)}{f'(b) - f'(a)},$$

the relation

$$\frac{bf'(b) - af'(a) - f(b) + f(a)}{f'(b) - f'(a)} = sa + (1 - s)b$$

can be cross-multiplied and rearranged to be converted into

$$\frac{f(b) - f(a)}{b - a} = (1 - s)f'(a) + sf'(b).$$

Since all the steps of the above proof are reversible, the assertion “if and only if” of the proposition follows.

The above proposition provides an indirect way of finding the point of intersection of two intersecting tangent lines to a differentiable function, once you know the slopes of the tangent lines at the two points, as seen in the following example.

EXAMPLE: Let $f(x) = e^x$, and consider the two points $A(a, f(a)) = (1, e)$, $B(b, f(b)) = (2, e^2)$ on the graph of this function. Here, $f'(1) = e$ and $f'(2) = e^2$. Noting that the graph of $f(x)$ obviously suggests that T_a and T_b intersect, and since

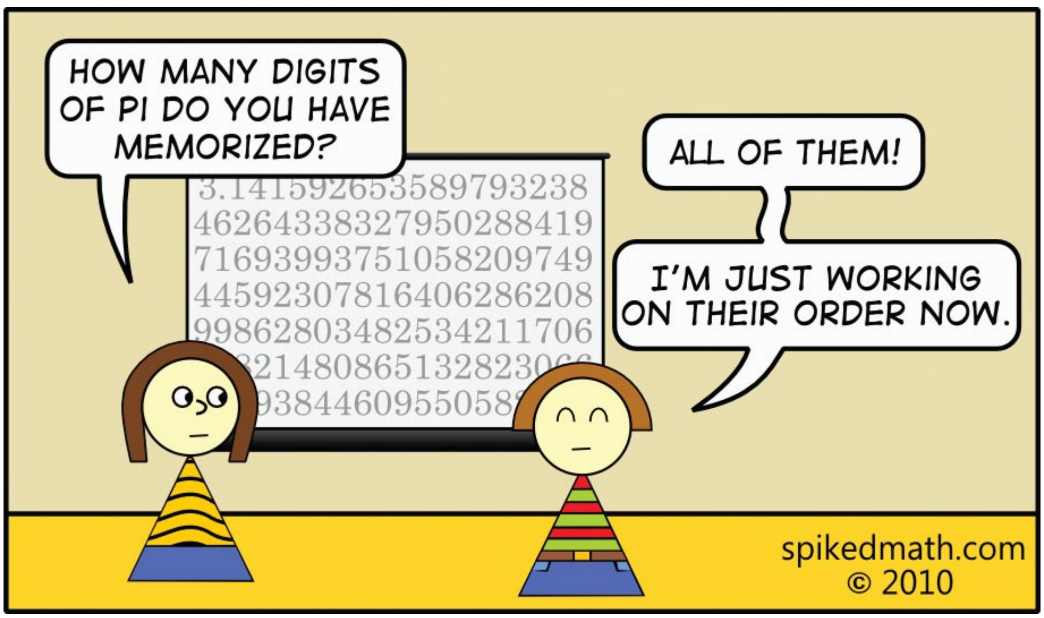
$$\frac{f(b) - f(a)}{b - a} = \frac{e^2 - e}{2 - 1} = e^2 - e = s \times f'(1) + (1 - s) \times f'(2) = s \times e + (1 - s) \times e^2$$

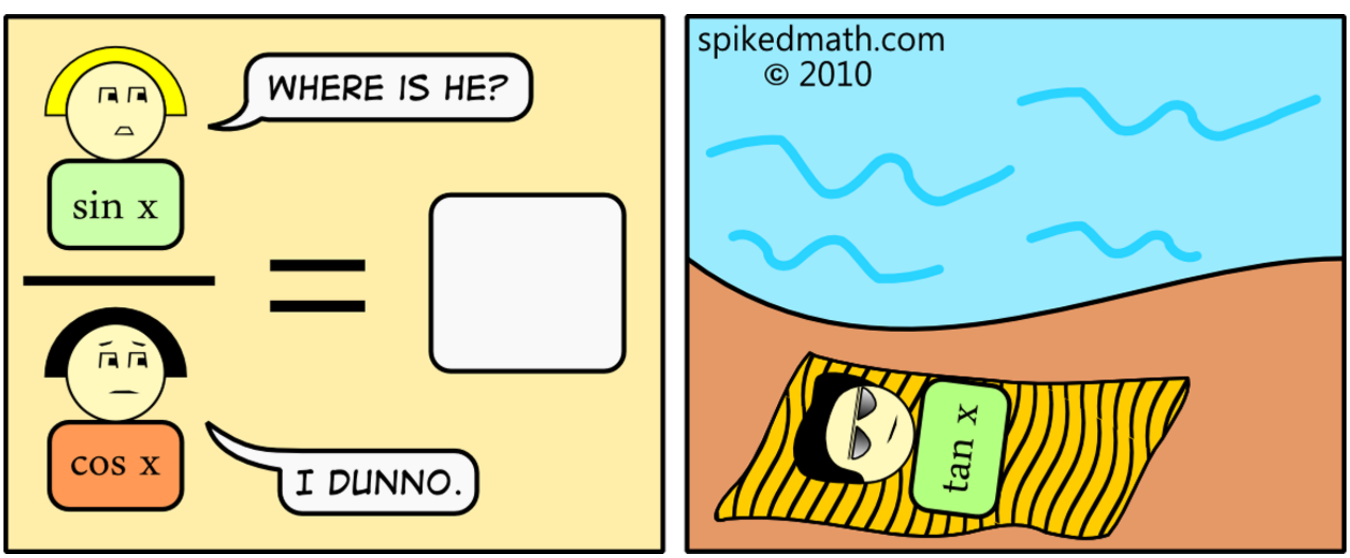
with $s = \dfrac{1}{e-1}$, the proposition implies the abscissa of the point of intersection of T_a and T_b should

be the ***skewed-corresponding*** combination $c = (1-s)\times e + s \times e^2 = \dfrac{e}{e-1}$.

Therefore $(e/(e-1),\ e^2/(e-1))$ would be the point of intersection of T_a and T_b. This fact can also be confirmed by directly solving the following system of linear equations.

$$\begin{cases} T_a: y = e(x-1)+e \\ T_b: y = e^2(x-2)+e^2 \end{cases}.$$





STAY IN CONTROL WITH MATH

Monique Chyba, Michael Andonian, Geoff Patterson, John Rader: University of Hawaii at Manoa, SUPER-M (A GK-12 program funded by the National Science Foundation)

1 INTRODUCTION

We live in an era of exciting scientific advances such as discovering new planets and black holes far away in the universe or gaining a better understanding of our own biological system. Unsurprisingly, mathematics plays a dominant role in almost all of them. Indeed, mathematics is the appropriate language and framework to formulate scientific questions and to analyze them. Beautiful mathematics is used every day by physicists, engineers, astronomers, biologists and other scientists. Almost the entire spectrum of mathematics can be found in the work of scientists. Let us focus here on one specific field: control theory. Control theory models, analyzes and synthesizes the behavior of dynamical systems. Those systems are described by sets of ordinary differential equations that include an additional parameter referred to as the 'control'. It can be viewed as the ship's wheel of the system in analogy to the navigation of a boat. A vast area of work takes place in optimal control theory. Indeed, since by using different controls we can achieve the same goal, optimization with respect to a given cost such as energy or time becomes a primary interest. Optimal control is an extension of the calculus of variations which can be viewed as an advanced version of the calculus taught at the high school level. The three specific examples described in the next sections provide concrete applications of control theory. Extremely sophisticated mathematics, ranging from differential geometry to computational mathematics, is involved in the study of each of these applications. On the other side, these applications are the perfect platform to push further our mathematical theories.

2 BIOLOGICAL MYSTERIES OF OUR BRAIN

From a single fertilized egg, all life on Earth takes its own unique shape and characteristics. One of the big questions in the field of biology is how, from a single cell, that the complex structures of the body can arise. Uninterrupted, a mass of cells would prefer a compact, spherical shape. From this mass, how do the organs and the body take shape?

In modeling the development of the brain from stem cells, it has been shown that the cells themselves produce certain chemicals, called growth factors, that diffuse in the extra-cellular space around the cells. Other types of cellular structures that are created by the cells, named fractones, absorb the growth factor from the surrounding environment, and once the concentration of growth factor has reached a given threshold they signal to an attached cell to undergo mitosis.

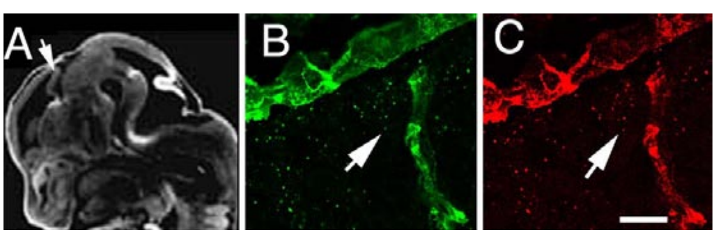


FIGURE 1: CAPTURE OF BROBLAST GROWTH FACTOR-2 (FGF-2) BY FRACTONES IN A MOUSE EMBRYO.

Modeling equations for this fundamental biological process have been developed using control theory. They are structured to capture the most significant aspects of the underlying biological process such as how growth factor diffuses or how the mass of cells takes shape postmitosis. The original question, though, remains: how does the model explain how the body takes shape? In this model, a user has the ability to control when and where a fractone will appear. The location of a fractone in the space dictates the direction in which new cells will eventually form, hence giving the ability to break away from the spherically symmetric mass of cells and create potentially any connected shape we want. Biologically, the body is programmed by the DNA to do this efficiently, hence for the model, there is the question of how to create a given cellular structure by placing the fractones optimally.

3 REACHING THE OUTER SPACE WORLD

The seemingly vast emptiness of space surrounding the Earth is not actually as empty as one may think. In fact, there are large amounts of rocks and other space materials that regularly pass by Earth on their journey across the universe. There is some evidence that suggests that frequently some of this debris gets caught by the Earth's gravitational pull and ends up in orbit around the Earth for months. These orbiting rocks can be called Natural Earth Satellites (NES). If we could obtain a sample of a NES, we could have access to rock material that has travelled from the far reaches of space, a perhaps unprecedented accomplishment. So, we pose the question: could we design a spacecraft mission to an NES?

To have any hope of accomplishing this, we need to study how objects move in space. First, any object in space attracts other objects in space with a force proportional to its mass. These attractions can be described using mathematical equations, and more specifically, can be described as a dynamical system using an area of calculus known as Differential Equations.

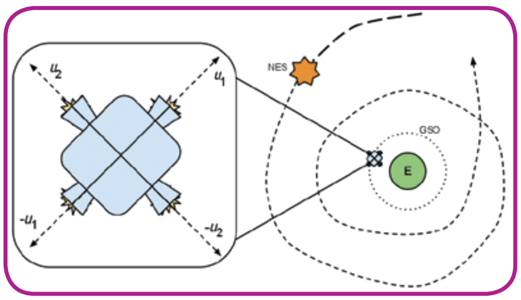
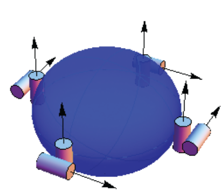


FIGURE 2: A TWO-DIMENSIONAL MODEL OF THE EARTH, OUR SPACECRAFT, AND AN NES ALONG WITH ITS TRAJECTORY. THE SPACECRAFT BEGINS ON THE GEOSTATIONARY ORBIT (GSO), AND THRUSTS IN THE POSITIVE OR NEGATIVE U1 OR U2 DIRECTIONS IN ORDER TO RENDEZVOUS WITH THE NES.

The gravities of the planets are not the only forces acting on our spacecraft. Naturally, we can also control the movement of the spacecraft with some type of thrusters. We would like the spacecraft to be able to steer itself to the NES automatically, which means we can rephrase our original question to be: How do we program the spacecraft to autonomously pilot itself to the NES? More specifically, when should the spacecraft fire which thrusters? We could also ask: which thrusters do we fire to get to the NES using as little fuel as possible? This type of problem is another example of an optimal control problem.

4 EXPLORING THE UNKNOWNS OF THE UNDERWATER WORLD

In the hostile waters of the earth lie scientific questions yet to be answered. As an attempt to study some of these questions, Autonomous Underwater Vehicles (AUVs) have allowed researchers to explore these underwater environments, too hostile and dangerous for man or manned vehicles. AUVs are basically robotic submersibles capable of navigating through treacherous environments, some with little to no human interaction. But AUVs sent into hazardous environments face the reality that there is a good chance they will become damaged. In such situations, it is not uncommon for the AUV to lose the ability to control any number of its thrusters. In which case, how can we be sure the AUV can continue its mission or even return home? This vital question can be answered through some beautiful mathematics. Using techniques from a branch of mathematics called differential geometry, we can describe precisely the motions capable for an AUV that has experienced thruster failure. From there, concatenating a series of permissible motions allows us to determine practical paths the underactuated AUV may follow. The techniques that have been developed have allowed us to simulate AUV missions in a variety of underactuated scenarios and environments, from surveying an underwater volcano, Loihi, to tranversing along the largest river in Colombia.



$$\nabla_{\gamma'}\gamma' = \mathbb{G}^{\#}P\left(\gamma\left(t\right)\right) + \mathbb{G}^{\#}\left(F_{drag}\left(\gamma'\left(t\right)\right)\right) + \sum_{i=1}^{6}\mathbb{I}_i^{-1}\left(\gamma\left(t\right)\right)\sigma_i\left(t\right)$$

FIGURE 3: GENERATED IMAGE BASED ON A REAL AUV. BELOW ARE THE GEOMETRIC EQUATIONS OF MOTION USED TO DESIGN EFFICIENT CONTROL STRATEGIES.

2010 MATH CHALLENGE SOLUTIONS

SOLUTIONS TO THE PROBLEMS PUBLISHED IN THE ISSUE 14 OF **PI IN THE SKY**

PROBLEM 1.

Find all the real pairs (x, y) such that

$$\log_3 x + \log_x 3 \leq 2\cos \pi y \qquad (*)$$

Solution:

We should have $x \in (0,1) \cup (1, \infty)$. If $x > 1$ then the Arithmetic Geometric Mean Inequality gives

$$\log_3 x + \log_x 3 \geq 2\sqrt{\log_3 x \cdot \log_x 3} = 2$$

Since $2\cos \pi y \leq 2, \forall y \in \mathbb{R}$ the inequality (*) holds if and only if simultaneously $\log_3 x + \log_x 3 = 2$ and $2\cos \pi y = 2$, hence $x = 3$ and $y = 2k$, where k is any integer.

If $x \in (0,1)$ then again, by using the Arithmetic Geometric Mean Inequality we obtain

$$(-\log_3 x) + (-\log_x 3) \geq 2\sqrt{(-\log_3 x) \cdot (-\log_x 3)} = 2$$

or

$$\log_3 x + \log_x 3 \leq -2$$

Since $2\cos \pi y \geq -2$ the given inequality holds for $\forall x \in (0,1)$ and $\forall y \in \mathbb{R}$. Therefore the requested pairs are $(3, 2k), k \in \mathbb{Z}$ and (x, y) where $0 < x < 1$ and $y \in \mathbb{R}$.

PROBLEM 2.

Determine all the triples (a, b, c) of integers such that $a^3 + b^3 + c^3 = 2011$

Solution:

For any integer k, we have $k^3 \equiv 0, \pm 1 \pmod 9$, hence $a^3 + b^3 + c^3$ is not congruent to 4 (mod 9). On the other hand since $2011 \equiv 4 \pmod 9$ we conclude that the equation does not have any integral solution.

PROBLEM 3.

In the decimal representation the number 2^{2010} has m digits while 5^{2010} has n digits. Find $m + n$.

Solution:

If the number A has p digits and $10^{p-1} < A < 10^p$ then $p - 1 < \log_{10} A < p$ and hence $p = \log_{10} A + \alpha$, for some $\alpha \in (0,1)$.

Thus

$$m = 2010\log_{10}2 + \mu, n = 2010\log_{10}5 + v, \mu, v \in (0,1).$$

and therefore

$$\begin{aligned} m + n &= 2010\log_{10}2 + \mu + 2010\log_{10}5 + v \\ &= 2010 + \mu + v = 2011 \end{aligned}$$

PROBLEM 4.

Find all the polynomials $P(x)$ with real coefficients such that $\sin P(x) = P(\sin x), x \in \mathbb{R}$

Solution:

We first remark that $P(x) = 0$ and $P(x) = \pm x$ are solutions of the problem. Let us prove that there is no other solution.

If $P(x) = ax + b, a \neq 0$ is a solution of the problem then $\sin(ax + b) = a\sin x + b, x \in \mathbb{R}$.

If we take $x = 0$ we get $\sin b = b$. Now if we take $x = \pi$ and then $x = \pi/2$ we get $a = \pm 1$ therefore we get two of the above already mentioned solutions.

Let us assume for a contradiction that there is a polynomial $P(x)$ of degree at least 2 that is a solution to the problem. We present two arguments showing this is impossible.

For this first, since P is a polynomial, it has only finitely many roots in $[-1, 1]$, say n roots. Then in any interval $[s, s + 1], P(\sin x)$ can have at most $2n$ roots since on an interval of length 1, $\sin x$ can take each value at most twice.

On the other hand, since $P(x)$ is of degree at least 2, we have $|P'(x)| \rightarrow \infty$ as $x \rightarrow \pm\infty$ ($P'(x)$, the derivative, being a polynomial of degree at least 1). In particular, there is an interval $[s, s + 1]$ on which $|P'(x)| > (2n + 1)\pi$.

By the mean value theorem, $|P(s + 1) - P(s)| > (2n + 1)\pi$. Hence the interval between $P(s)$ and $P(s + 1)$ contains at least $2n + 1$ multiples of π. Using the intermediate value theorem $P(x)$ takes on at least $2n + 1$ values that are multiples of π on the interval $[s, s + 1]$, guaranteeing that $\sin P(x)$ has at least $2n + 1$ roots on this interval.

Since $P(\sin x)$ and $\sin(P(x))$ have different numbers of roots on $[s, s + 1]$, they cannot be equal.

The second argument is as follows (assuming again that P is a polynomial of degree at least 2). We must have

$$\sin(P(x + 2\pi)) = P(\sin(x + 2\pi)) = P(\sin x) = \sin P(x), \forall x \in \mathbb{R}.$$

Hence, for all x we can find an integer $k(x)$ or $h(x)$ such that

$$P(x + 2\pi) - P(x) = 2k(x)\pi \text{ or } P(x + 2\pi) + P(x) = (2h(x) + 1)\pi$$

For each pair of integers (k, h) let us define the following sets of real numbers:

$$A_{(k-h)} = \{x \in \mathbb{R}; P(x + 2\pi) - P(x) = 2k\pi \text{ or } P(x + 2\pi) + P(x) = (2h + 1)\pi\}.$$

Since $\mathbb{R} = \bigcup_{(k,h)\in Z\times Z} A_{(k,h)}$ we conclude that at least one set $A_{(k,h)}$ should be infinite, otherwise $\bigcup_{(k,h)\in Z\times Z} A_{(k,h)}$ would be a countable set while $\mathbb{R}$ is not.

If $A_{(k,h)}$ has infinitely many elements then at least one of the two equations $P(x+2\pi) - P(x) = 2k\pi$ or $P(x+2\pi) + P(x) = (2h+1)\pi$ has infinitely many roots. This is impossible since neither of the polynomials $P(x+2\pi) - P(x)$ or $P(x+2\pi) + P(x)$ is constant.

PROBLEM 5.

The interior of an equilateral triangle of side length 1 is covered by eight circles of the same radius r.

Prove that $r \geq \frac{1}{7}$.

Solution:

If we divide each side of the triangle in seven equal parts and draw parallel lines through all these points to the sides of the triangle we get 49 equilateral triangles of side $^1/_7$. The number of vertices of the configuration that is obtained is 1 + 2 + ... + 8 = 36. Since $36 > 4.$ we obtain that there are five vertices which should stay inside (or on) at least one circle of the eight circles. Since the smallest radius of a circle that contains five points of the configuration is $\frac{1}{7}$ we obtain that $r \geq \frac{1}{7}$.

PROBLEM 6.

Prove that in a convex hexagon of area S there exist three consecutive vertices A, B and C such that

$$Area(ABC) \leq \frac{S}{6}$$

Solution:

If the diagonals AD, BE and CF of the hexagon $ABCDEF$ intercept at the point O then the area of at least one of the quadrilaterals $OABC$, $OCDE$ or $OEFA$ is $\leq \frac{S}{3}$. If for example Area($OABC$) $\leq \frac{S}{3}$ then at least one of the triangle OAB and OBC has area $\leq \frac{S}{6}$. For example, if we assume that Area(OAB) $\leq \frac{S}{6}$, then we immediately conclude that the area of the triangle FAB or CAB is $\leq \text{Area}(OAB) \leq \frac{S}{6}$, as requested.

If the diagonals $AD \cap FC = \{L\}, AD \cap BE = \{M\}$ and $BE \cap CF = \{N\}$ then

$$\text{Area}(ABNF) + \text{Area}(BCDM) + \text{Area}(EDLF) \leq \text{Area}(ABCDEF)$$

therefore the area of at least one of the quadrilaterals $ABNF$, $BCDM$ or $EDLF$ is $\leq \frac{S}{3}$. For example, if we assume that Area($EDLF$) $\leq \frac{S}{3}$ then the area of at least one of the triangles FEL or EDL is $\leq \frac{S}{6}$. For example if we assume that Area(LEF) $\leq \frac{S}{6}$, we find that at least one of the triangles AFE and DFE has area less or equal to the area of the triangle LEF (since the distance from L to FE is greater or equal to the minimum of the distances from A and D to FE) and consequently there exists at least one triangle of area $\leq \frac{S}{6}$, made with three consecutive vertices of the hexagon.

2011 MATH CHALLENGES

PROBLEM 1

The decimal part of $x = 0.2499...975$ contains 2007 consecutive 9's. Find the first 2011 decimal digits of $\sqrt{x}$.

PROBLEM 2

How many pairs of positive integers sum to 2011 and have a product that is a multiple of 2011?

PROBLEM 3

Find all triplets (x, y, z) of real numbers such that $x + \sqrt[3]{x} = 2y, y + \sqrt[3]{y} = 2z$ and $z + \sqrt[3]{z} = 2x$.

PROBLEM 4

Let a be a fixed integer. Find all the functions $f: \mathbb{Z} \to \mathbb{N}$ such that for any $n \in \mathbb{Z}$

$$f(n-a) + f(n+a) \leq 2f(n)$$

(Here $\mathbb{Z}$ denotes the set of all integers while $\mathbb{N}$ denotes the set of positive integers).

PROBLEM 5

Given a set of $2n$ distinct points in the plane, prove that there exist n line segments joining pairs of these points such that no two of them intersect.

PROBLEM 6

Let $\triangle ABC$ be an equilateral triangle, and suppose EF is parallel to BC, where $E \in (AB)$ and $F \in (AC)$. Let O be the centroid of $\triangle AEF$ and M be the midpoint of (EC). Find the angle $\widehat{OBM}$.

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