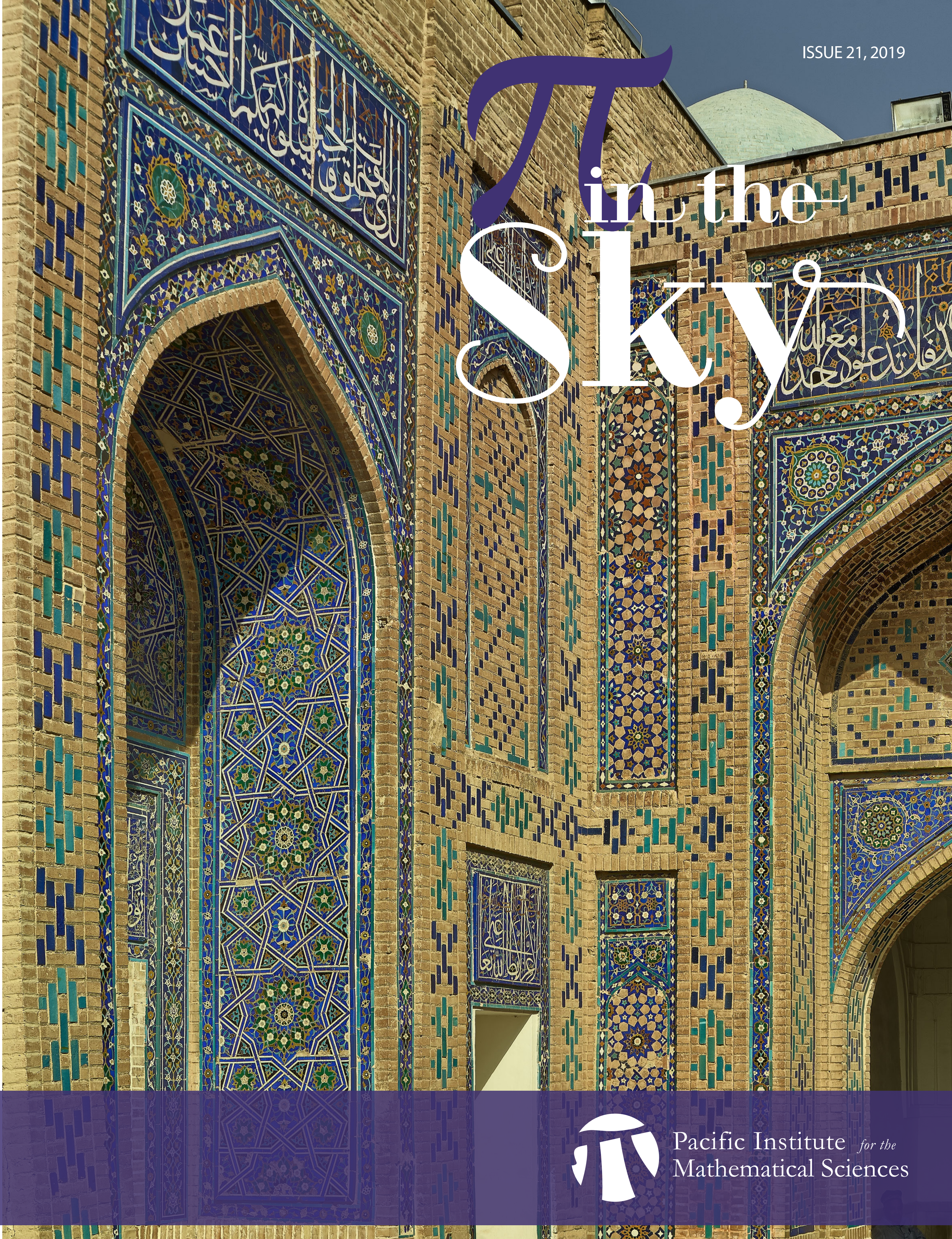
ISSUE 21, 2019

π in the Sky



Pacific Institute for the Mathematical Sciences

Pi in the Sky

Issue 21, 2019

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Submission Information

For details on submitting articles for our next edition of Pi in the Sky, please see: http://www.pims.math.ca/resources/publications/pi-sky

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Pi in the Sky is aimed primarily at high school students and teachers, with the main goal of providing a cultural context/landscape for mathematics. It has a natural extension to junior high school students and undergraduates, and articles may also put curriculum topics in a different perspective.

Welcome to Pi in the Sky!

The Pacific Institute for the Mathematical Sciences (PIMS) sponsors and coordinates a wide assortment of educational activities for the K-12 level, as well as for undergraduate and graduate students and members of underrepresented groups. PIMS is dedicated to increasing public awareness of the importance of mathematics in the world around us. We want young people to see that mathematics is a subject that opens doors to more than just careers in science. Many different and exciting fields in industry are eager to recruit people who are well prepared in this subject.

PIMS believes that training the next generation of mathematical scientists and promoting diversity within mathematics cannot begin too early. We believe numeracy is an integral part of development and learning.

For more information on our education programs, please contact one of our hardworking Education Coordinators.

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Solutions to Math Challenges at the end of this issue will be published Pi in the Sky Issue 22. See details on page 24 for your chance to win $100!

A Note on the Cover

The cover image is the Tuman Aqa Mausoleum in the Shah-i-Zinda Complex in Samarkand, Uzbekistan. It was taken by Peter Lu, a Harvard-based physics researcher. Ancient Islamic buildings are often decorated with beautiful tile patterns as in the cover image. Lu, while he was working on his Ph.D. discovered that these patterns are, in many cases, quasi-crystals.

Crystals are periodically repeating patterns (such as squares making up a grid, or interlocking hexagons). These patterns may have rotational symmetry (they may look the same if the entire pattern is rotated) but as has been known since the 19th century, the only possible rotation amounts are by 60, 90, 120 or 180 degrees (that is 2-fold, 3-fold, 4-fold and 6-fold symmetry). In the 1980's, parallel developments in mathematics and materials science showed the existence of a mathematical basis for materials with 5-fold and 10-fold symmetry. These fascinating structures are not periodically repeating like standard crystals, but they share many properties with crystals.

Peter Lu's remarkable discovery was that in spite of the fact that quasi-crystals were only identified and studied in the late 20th Century, their development was anticipated in medieval Islamic architecture in the form of the tilings that adorn some important ancient Islamic buildings. Lu has taken stunning photographs of these buildings, and hypothesized ways in which labourers constructed the tilings - even managing to identify places in which the labourers made errors in following the designs.

For more details, see Peter Lu's website:
peterlu.org

For an article in Discover Magazine on his work:
discovermagazine.com/2008/jan/math-breakthrough-spotted-on-mosques

MATHEMATICAL CUT-AND-PASTE:
An Introduction to the Topology of Surfaces

BY MAIA AVERETT

Associate Professor of Mathematics and Head of Mathematics and Computer Science, Mills College.

Fig. 1: A donut Earth? Why not?.

A mathematician named Klein
Thought the Möbius band was divine.
Said he, "If you glue
The edges of two,
You'll get a weird bottle like mine."

-Anonymous

LET US BEGIN WITH A SIMPLE QUESTION: What shape is the earth? Round, you say? Ok, but round like what? Like a pancake? Round like a donut? Like a soft pretzel? Some other tasty, carbohydrate- laden treat? No, no—it's round like a soccer ball. But how do you know? ...Really, how do you know? Perhaps you feel sure because you've seen photos of the earth from space. Well, people figured out that the earth is round long before we figured out how to build rocket ships (or cameras, for that matter!).

Scientists as far back as the ancient Greeks theorized that the earth is round. Although they offered no substantive proof of their theories, Pythagoras, Plato, and Aristotle were all supporters of the spherical earth theory, mostly based on the curved horizon one sees at sea. Surely this suggests that the earth is not flat like a pancake, but how can we know that the earth isn't some other round shape, like a donut, for example?

If we were to walk around the entire earth, then we can come up with plenty of reasons that it's not shaped like a donut.

The most obvious, perhaps, is that if the earth were a donut, there would be some places where we could stand and look directly up into the sky and see more of the earth! Also, there would be places where the curve of the horizon would be upwards instead of downwards. But how can we really, truly know that the shape is that of a ball and not some other strange shape that we haven't yet thought up? As a thought experiment, pretend for a moment that you are locked in a room with thousands and thousands of maps of various places on Earth. Suppose you have enough maps so that you have several for every point on the globe. Could you determine the shape of the earth? Yes! You need only paste together the maps along their overlaps.

This basic idea is exactly the idea that underlies the way mathematicians think about surfaces. Roughly speaking, a *surface* is a space in which every point has a neighborhood that "looks like" a two-dimensional disk (i.e. the interior of a circle, say $\{x, y \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$.)

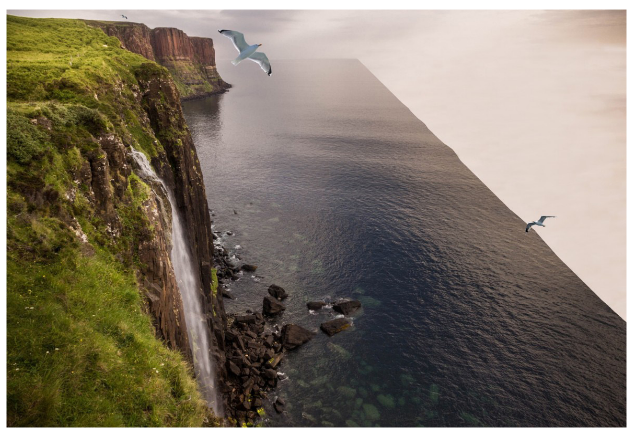


Fig. 2: A surface with boundary.

A sphere is an example of a surface, as is the surface of a donut, which mathematicians call a torus. Some of our everyday, natural notions of surfaces don't quite fit this definition since they have edges, or places where you could fall off if you weren't careful! Mathematically, these are *surfaces with boundary*: spaces in which every point has a neighborhood that looks like either a two-dimensional disk or half of a two dimensional disk (i.e. ... $\{x, y \in \mathbb{R}^2 \mid x^2 + y^2 < 1, y \geq 0\}$). The old circular model of the earth where you can sail off the edge is an example of a surface with boundary. Another example is a cylinder without a top and bottom. Now that we know what a surface is, let's to try to figure out what kinds of surfaces are out there. Here, we're going to examine this question from a *topological* point of view—we'll be interested in the general *shape* of the surface, not in its *size*. Although the geometric notions of size and distance are quite important in reality, topologists seek to understand the coarser structure of surfaces as a first approximation to understanding their shape.

For example, from a topological point of view, a sphere is a sphere, it doesn't matter how large or small the radius is. To this end, we will allow ourselves to deform and manipulate surfaces as if they were made of rubber sheets: we'll consider two surfaces to be the same if we can stretch, shrink, twist, push, or wriggle one surface around until it looks like the other surface. But we will have to be nice in our deformations: topologists aren't so violent as to create holes or break or tear any part of our surface. So, an apple would be considered the same as a pear, doesn't matter if it has a big lump on one end. A flat circular disc is the same as the upper half of the surface of a sphere, even though the latter is stretched and curvy. The classic joke in this vein is that a topologist can't tell the difference between a coffee cup and a donut. If we had a flexible enough donut, we could make a dent in it and enlarge that dent to be the container of the coffee cup, while smooshing (certainly a technical topological term) the rest of the donut down in to the handle of the coffee cup.

Let's begin by trying to make a list of surfaces that we know. What surfaces can you think of? The first one that comes to mind is the surface of the earth: it's a sphere. (Note here that we're only talking about the surface of the earth, not all the dirt, water, oil, and molten rock that make up its insides! Just the surface—like a balloon.) Another surface that comes up a lot is the *torus*, which is shaped like an innertube. For the most part here, we're going to restrict our investigation to compact (which means *small* in the loose sense that they can be made up of finitely many disks patched together) and connected (made of one piece, i.e. you can walk from one point to every other point on the surface without jumping). We will see some examples of surfaces with boundary because they are surfaces that you may be familiar with. As mentioned before, a cylinder without a top or bottom is a surface with boundary. A Möbius strip is a surface with boundary.

Drawing surfaces on paper or on the blackboard is difficult. One needs quite an artistic hand to convey the shape of an object that lives in our three-dimensional world accurately on two-dimensional paper. However, we'll see that it's easy and quite convenient to record cut-and-paste instructions for assembling surfaces with a simple diagram on a flat piece of paper.

We take our inspiration from maps of the world. In a typical world map, the globe is split open and stretched a bit so it can be drawn flat. We all understand that if we walk out the right side of the map, we come in through the left side at the same height. This is a pretty useful idea!

We can imagine a seam on a globe that represents this edge. We can think of taking the map and gluing up the left and right edge to return to our picture of the globe.
There is one slight dishonesty in the typical world

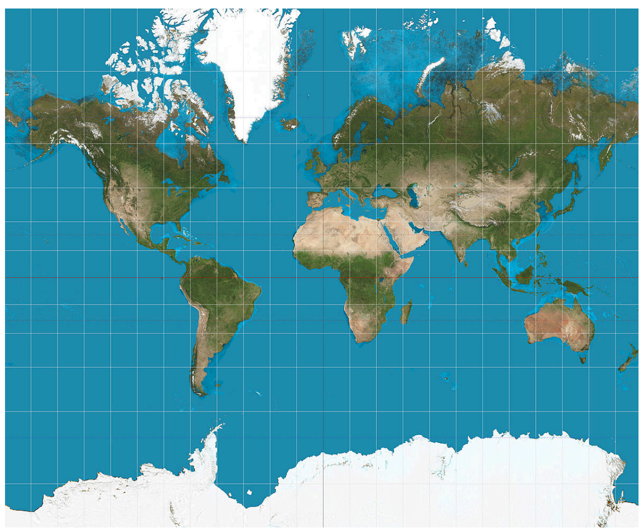
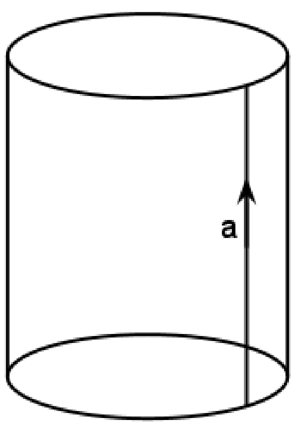
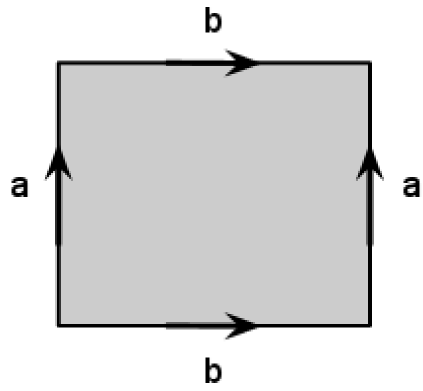


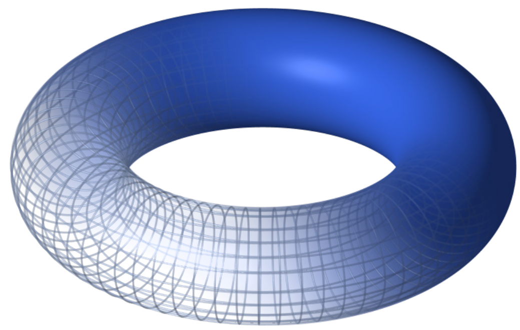
Fig. 3: Map of the world [11]

map: the representations of very northerly and southerly parts of the earth aren't very accurate. They're much bigger on the map than they are in reality! In fact, the entire line at the top edge of the map really represents just a single point on the globe, the north pole. Similarly, for the bottom edge and the south pole. We can make a more honest map by shrinking these edges down so that we have one point at the top and one point at the bottom, representing the north and south poles, respectively. Then our resulting picture is a circle! It has the same properties with respect to walking out through the right edge and coming back in through the left. We can record this information by drawing arrows on the boundary of the circle to indicate how we are to glue up the picture to create a globe. It's a lovely picture: if we glue up one semicircular edge of a circle to the other semicircular edge (without twisting!) then the resulting surface is a sphere. Let's look at some more examples of how this works.

Example 1 (The cylinder). We can create a cylinder by using a piece of paper and gluing the ends together. Thus we can write down instructions for making a cylinder by drawing a square and labeling a pair of opposite edges with a little arrow that indicates gluing them together.







Example 2 (The torus). The diagram above [8] represents a gluing diagram for the torus. To see this, first imagine bringing two of the edges together to form a cylinder. Since the circles at the top and bottom of the cylinder are to be glued together, we can imagine stretching the cylinder around and gluing them to obtain a surface that looks like the surface of a donut. Now, let's practice thinking about how walking around on the surface is represented on the diagram. If we walk out the left edge, we come back in the right edge at the same height. Similarly, if we walk out the top, we come in the bottom at the same left-right position. It's like PacMan!

Exercise 1. Imagine you are a little two-dimensional bug living inside the square diagram for the torus above. You decide to go for a walk. Trace your path. Be sure to exit some of the sides of the square and be careful about where you come back in! Do this several times. Draw some torus gluing diagrams of your own and practice some more.

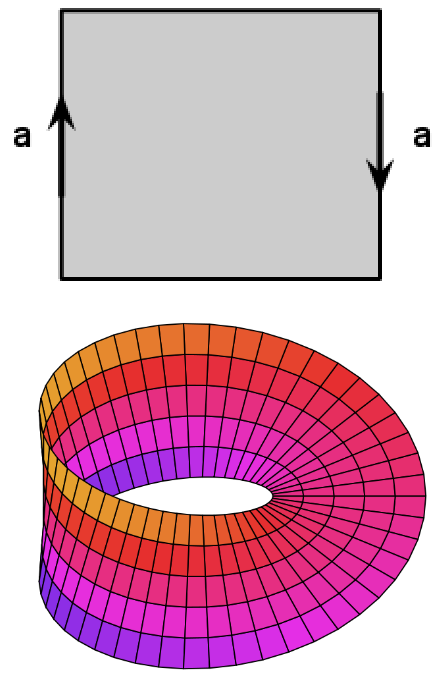


Fig. 4: Möbius strip[12]

Example 3 (The Möbius Strip). What happens if we start with a square and identify a pair of opposite edges, but this time in opposite directions? The resulting surface is a Möbius strip!

Exercise 2. A cylinder has two boundary circles. How many boundary circles does a Möbius strip have?

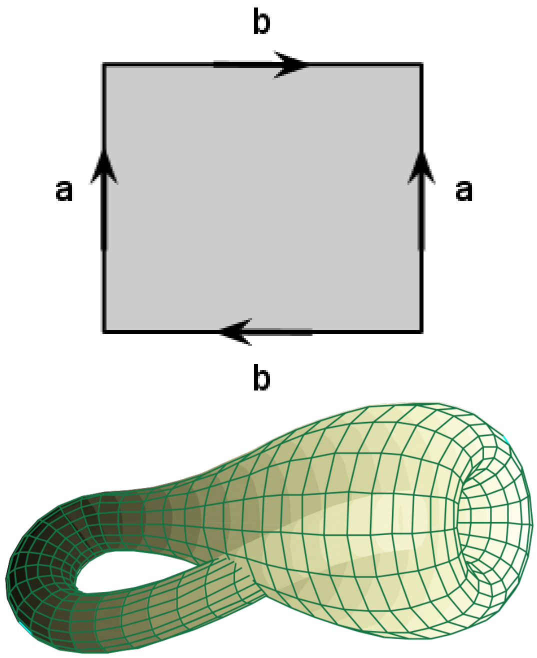


Fig. 5: Klein bottle [9]

Example 4 (Klein bottle) (Fig 5). What happens if we reverse the direction that we glue one of the pairs of edges in the diagram that we had for the torus? We can begin by again gluing up the edges that match up to create a cylinder. But now if we try to stretch it out and glue the boundary circles together, we see that the arrows don't match up like they did for the torus! We can't just glue the circles together because our gluing rule says that the arrows must match up.

The only way to imagine this is to imagine pulling one end of the cylinder through the surface of the cylinder and matching up with our circle from the inside. The resulting representation of the surface doesn't look like a surface, but it really is! It's funny appearance is just a consequence of the way we had to realize it in our three-dimensional world.

Exercise 3. Imagine you are a little two-dimensional bug living inside the square diagram for the Klein bottle above. You decide to go for a walk. Trace your path.

Be sure to exit some of the sides of the square and be careful about where you come back in! Do this several times. Draw some Klein bottle gluing diagrams of your own and practice some more!

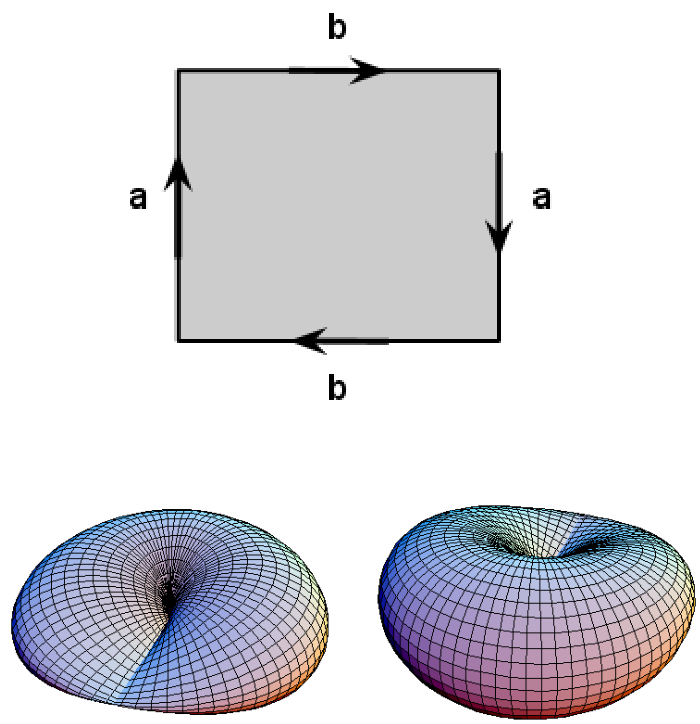


Fig. 6: Projective plane [10]

Example 5 (The projective plane) (Fig 6). What happens if we reverse not just one of the pairs, but *both* of the pairs of edges in our diagram for the torus? The resulting surface is called the *projective plane* and it is denoted $\mathbb{R}P^2$. It's hard to imagine what this surface looks like, but our square diagram will allow us to work with it easily!

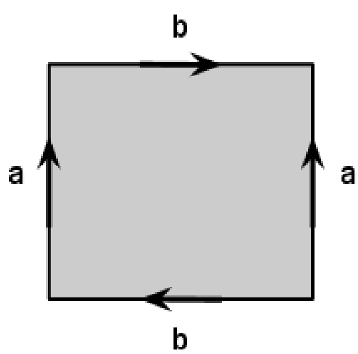
Exercise 4. Imagine you are a little two- dimensional bug living inside the square diagram for $\mathbb{R}P^2$ above. You decide to go for a walk. Trace your path. Be sure to exit some of the sides of the square and be careful about where you come back in! Do this several times. Draw some $\mathbb{R}P^2$ gluing diagrams of your own and practice some more!

Definition 1. A *gluing diagram* for a polygon is an assignment of a letter and an arrow to each edge of the polygon.

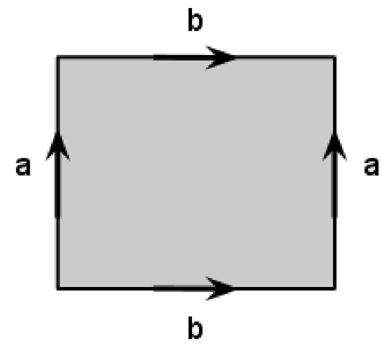
With this general definition, not every gluing diagram represents a surface. For example, if three edges are labeled with the same letter, then these glue up to give something whose cross section looks like ! However, if we assume that the edges are always glued in pairs, then the resulting pasted up object will always be a surface. Here's why. It's clear that every point in the interior of the polygon has a neighborhood that looks like a disk. A point on one of the edges but not on a corner has a neighborhood that looks like a disk if we think about the corresponding point on the edge that it's glued to and draw half- disks around each of them. A point on one of the corners can similarly be given a neighborhood that looks like a disk.

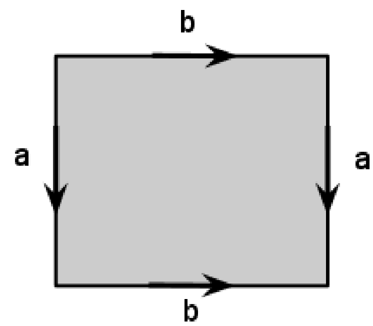
Example 6. The squares that we thought about above for the cylinder, the torus, the Klein bottle, the Möbius strip, and $\mathbb{R}P^2$ are gluing diagrams for these surfaces.

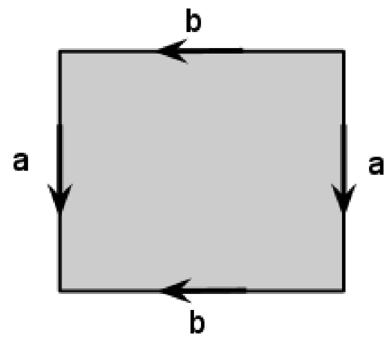
Exercise 5. What surface is represented by the gluing diagram below?



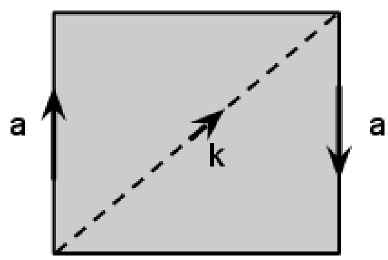
There might be many different diagrams that represent the same surface. For example, we could draw the diagram for the torus in the following ways (and this isn't even remotely all of them!). The important thing for a square to represent the torus is that opposite edges are identified without twists.



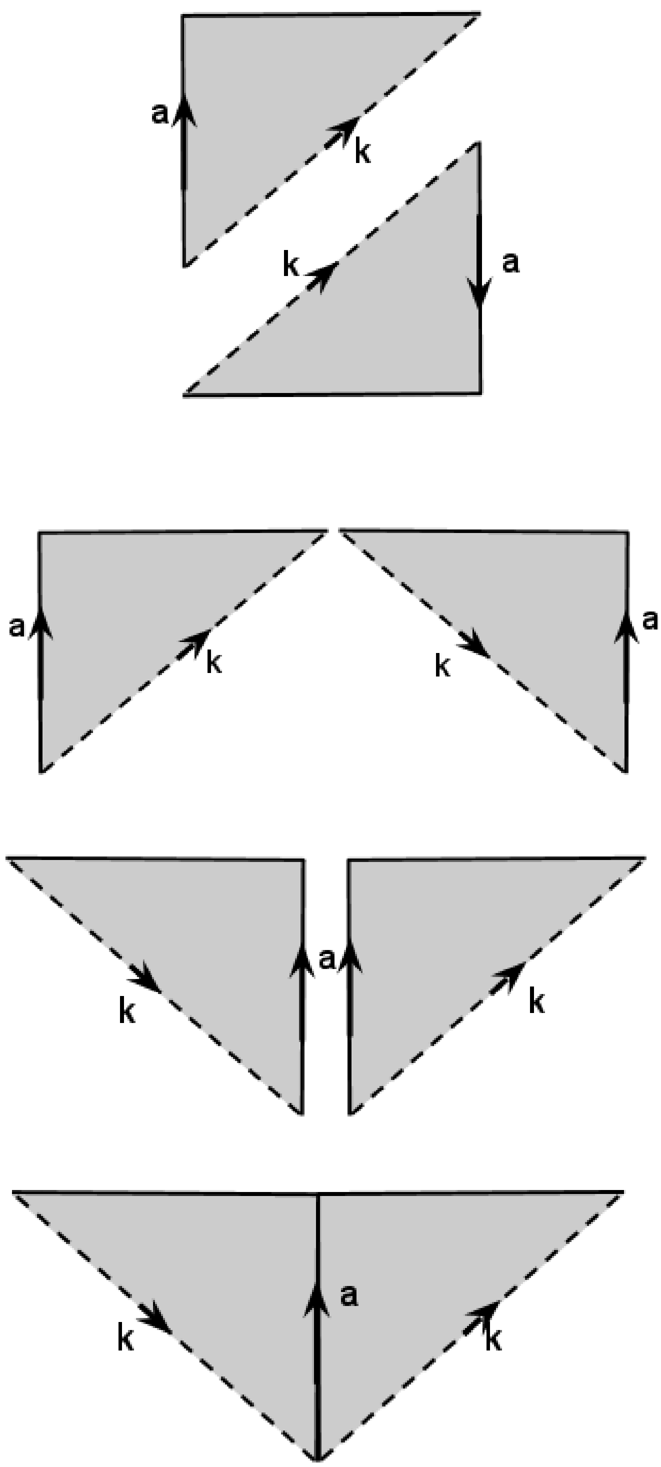




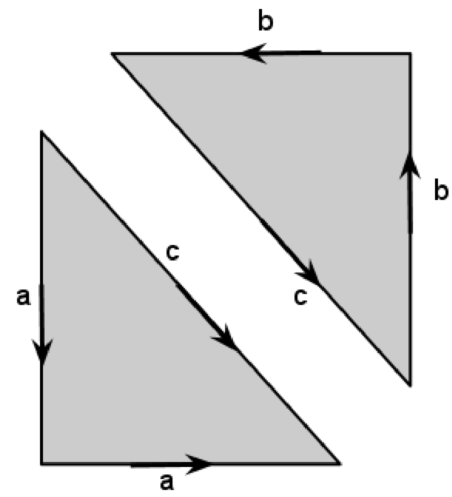
One technique for showing that two gluing diagrams represent the same surface is to take one of the diagrams, cut it, and reglue it (possibly repeatedly) until it looks like the other.

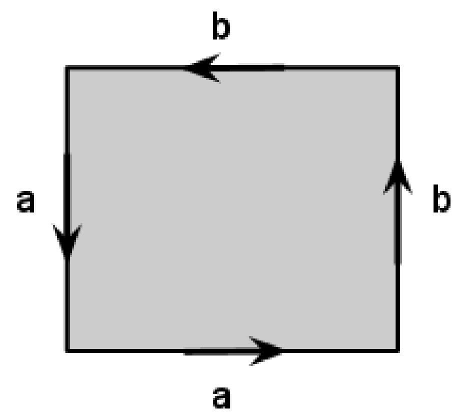


Example 7 (A Klein bottle made from two Möbius strips). In this example, we'll show that gluing two Möbius strips together along their boundary circles results in the Klein bottle. This explains the limerick at the beginning of these notes! First, we'll cut and rearrange the gluing diagram for the Möbius strip so that the boundary circle is displayed in one continuous piece.



Now we can see that the top edge of the triangle is the boundary of the Möbius strip, so this makes it easier to take two copies of the Möbius strip (in its new gluing diagram) and glue them together along their boundary circles (the boundary circles are labeled c in the diagram below on the left).



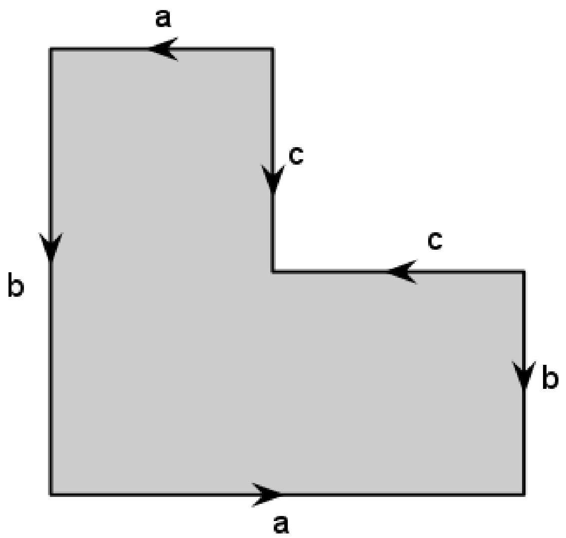


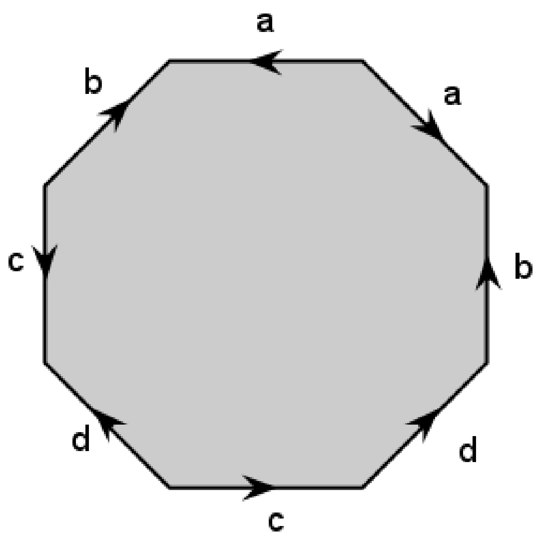
Hrmm... This doesn't quite look like our standard diagram for the Klein bottle! Your job in the next problem is to figure out how to cut it and rearrange the pieces so that it looks like the standard diagram.

Problem 1. Use cutting and regluing techniques to show that the gluing square above right represents the Klein bottle. Hint: Cut along a diagonal.

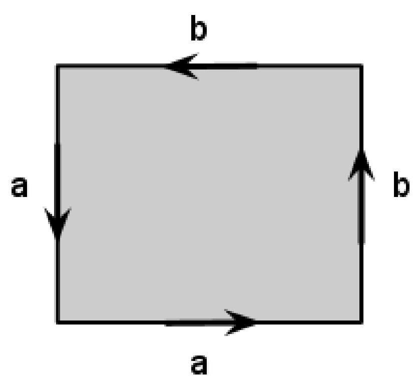
Problem 2. What surface results from gluing a disk to the boundary circle of a Möbius band?

Problem 3. Which of the following diagrams represent equivalent surfaces? (Note that each diagram represents its own surface. It is not intended that you glue all the a's together, etc, but only the ones on that specific diagram.)



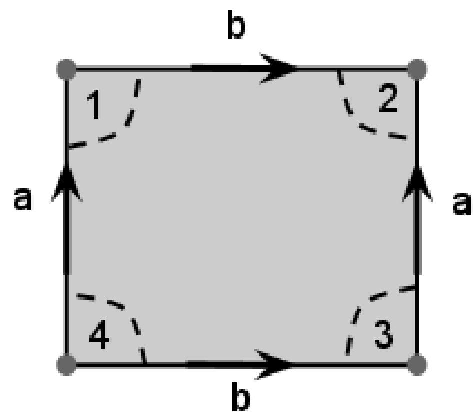


In a



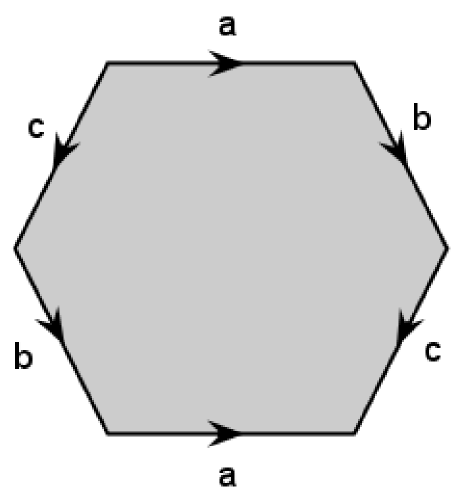
gluing diagram, we identify the edges of a polygon. This means that sometimes, the corners of our polygon are not distinct points on the surface it represents.

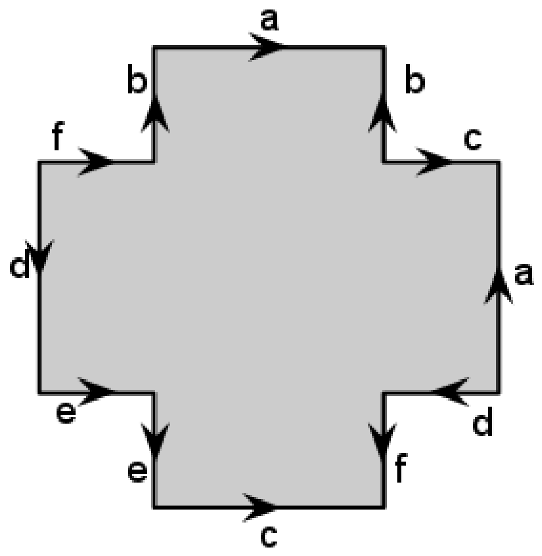
Problem 4. In the standard gluing diagram for the torus, all four corners represent the same point in on the surface of the torus. Cutting out a disk around this point is the same as cutting out the corners in the gluing diagram. Paste together the corners 1, 2, 3, and 4 so they form a disk. Do the same for a Klein bottle. What happens for $\mathbb{R}P^2$?



Exercise 6. Which corners in the standard square diagram for the Klein bottle represent distinct points in the surface? What about in the standard square for $\mathbb{R}P^2$?

Exercise 7. In each of the following diagrams, identify which corners represent the same point and which are distinct.





Problem 5. Since we are topologists, we don't care so much whether lines are straight or curved. We could also think about gluing diagrams that result from dividing a circle into subsegments (edges) and assigning letters and arrows to these edges. Our example of the circular world map is a gluing diagram for the sphere S^2 as a circle divided into two edges. Find a similar diagram for $\mathbb{R}P^2$.

One way to record the gluing is by writing down a word that describes what letters we see when we walk around the edges of the gluing diagram. Begin at one corner of the diagram and walk around the perimeter of the diagram. When we walk along an edge labeled with *a* letter, say *a'* in the same direction as its assigned arrow, we write that letter. If we walk along an edge labeled with a letter, say *a*, but in the opposite direction of its assigned arrow, we write down *a'*. The string of letters contains the same information as the gluing diagram, so long as we remember the code that translates between the words and the gluing diagram.

Exercise 8. Draw the gluing diagrams associated with the following words: *abab, abca'b'c', aba'b, ba'ba', ab'ab, bacc'b'a*.

Problem 6. Do any of the words in the previous exercise represent the same surface?

Problem 7. Consider gluing diagrams for a square that glue together pairs of edges. Let's use the letters *a* and *b* to denote the pairs of edges. How many are there? *Hint: To count them, you need to keep track of the letter of each edge and also its direction.*

Use the idea above of walking around the edge and recording the word you walk along. So, this is really a question that asks: how many four-letter words are that use the letters *a*, *a'*, *b*, *b'* such that both *a* and *b* appear exactly twice (where twice means with or without the decoration *'*, e.g. you could have a and *a*, or *a* and *a'*, or *a'* and *a'* in your list, but you cannot have a appearing only once or three times).

Now back to our initial question of trying to list all surfaces that there are. We might start by trying to count the different surfaces represented by these gluing diagrams. The number we just arrived at is certainly too large.

For example, if one diagram can be obtained from another by rotating it a quarter turn to the right, then these must represent the same surface. Similarly, if one diagram can be obtained from another by flipping the square over, they also must represent the same surface. By rotating and flipping our diagrams, we can reduce to the case where the left edge of the square is labelled with *a* and the arrow points up.

Problem 8. Now that we've determined that we can reduce to the case where the left edge of the square is labelled with a and with an upward pointing arrow, try to make a complete list of gluing diagrams that doesn't have any "obvious" repeats. By "obvious," I mean there isn't a sequence of rotations and a flip that will take one diagram on your list to another. Can you identify any of the diagrams as surfaces that we know?

Problem 9. Two of our diagrams turn out to represent the Klein bottle and two represent the projective plane $\mathbb{R}P^2$. Find a way to cut and paste the non-standard diagrams of the Klein bottle and $\mathbb{R}P^2$ so that they look like the standard ones.

You've just made a list of all the surfaces one can represent using a square, a lovely accomplishment! I hope you have enjoyed this brief journey into the twisted world of the topology of surfaces.

You're already quite close to having all the tools necessary to make a list of all possible surfaces. If you'd like to do so, a good jumping off point is to look up *Classification of Surfaces* online or in one of the texts referenced below.

ACKNOWLEDGEMENTS

I would like to thank Zvezdelina Stankova for inviting me to give a session at the Berkeley Math Circle years ago, as it was this session that inspired the writing of this article. I would also like to thank Greg Egan for the image of the torus Earth in Fig. 1. He was inspired to make the image by a MathOverflow question [6] asked by Michael Hardy.

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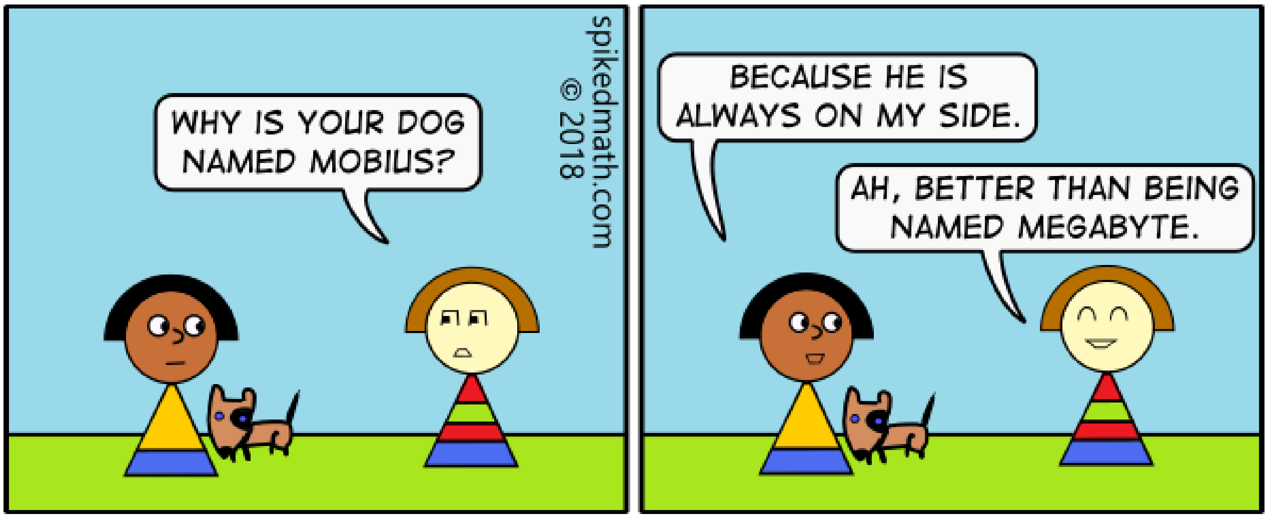


Geometric Probability and Irrational Numbers

BY KIMBERLY HOU
Kimberly wrote this when she was a high school student at BASIS Independent Silicon Valley. She is currently studying mathematics at Princeton.

INTRODUCTION

The fields of geometry and probability seem completely different. One focuses on the shape, size, and relative positions of figures while the other measures the relative likelihood that an event will occur. Yet, it is possible to measure a fundamentally geometric quantity using probability.

Pi is defined as the ratio of the circumference of a circle to its diameter. In 1777, Georges-Louis Leclerc, Comte de Buffon proposed a question that was one of the first in the field of geometric probability. Buffon's needle problem asks what the probability would be that a randomly thrown needle of length l will intersect one of infinitely many parallel lines of unit distance apart on a plane. Using the result for this problem, it is possible to create a method to approximate pi with surprising accuracy. In this paper, I will give another proof of this method.

FORMALIZATION

There are infinitely many horizontal parallel lines of unit distance apart on the plane. One will throw infinitely many needles of length l (where $l < 1$) onto this plane. I will calculate the probability of a needle intersecting any of the lines.

Below is a list of the variables used in the proof of this method:

h: the distance from the lower end of the needle to the parallel line immediate above it.

θ: the counter-clockwise angle between the needle and the line passing through the lower end of the needle and parallel to the parallel lines already on the plane.

PROOF

For a given l, whether or not the needle intersects a line is dependent on h and θ. For example, if the needle is parallel to the lines, it is impossible for it to intersect any lines. Yet if the needle is perpendicular to the lines, it is far more likely for it to intersect the lines. At the same time, if the needle is very short and the ends are far away from the parallel lines, it is again unlikely for it to intersect.

Now let's see how I calculate the probability that the needle intersects any of the parallel lines. Referring to Figure 1, one can see that the needle intersects a line if and only if the projection of the needle along the vertical direction ($l\sin\theta$) is greater than h, i.e. $h \le l\sin\theta$.

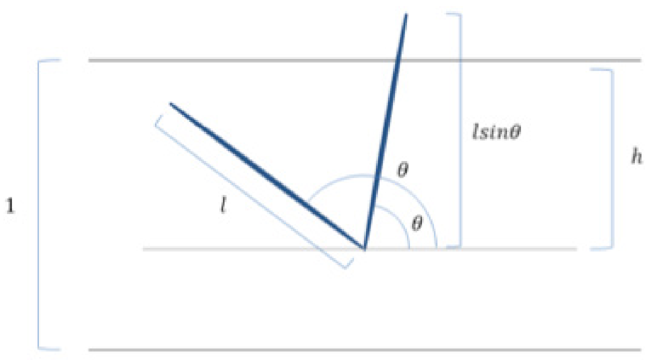


Figure 1

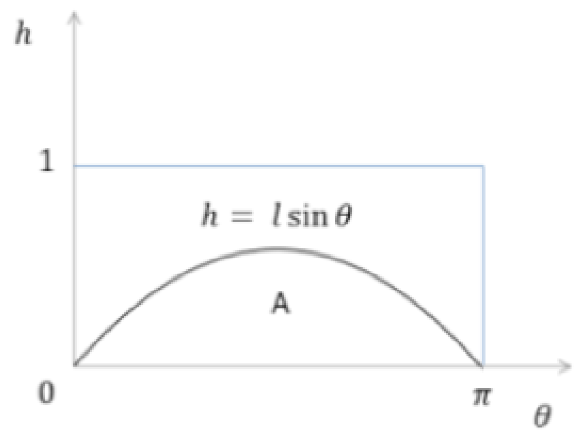


Figure 2

One can see that the possible range for θ is between 0 and π, and for h is between 0 and 1. This is shown in the graph in Figure 2. I denote A to be the region below the curve for the function $h(\theta) = l sin\theta$ where θ is between 0 and π. This represents when the needle intersects one of the parallel lines.

The area A, which includes all possible values of h and θ such that a needle interests a line, is calculated by the integral [1]

$$\int_0^\pi lsin\theta d\theta$$

The total area of the rectangle, which represents the total possible region for h and θ, is π. Hence the proportion can be calculated by the following integral

$$\frac{1}{\pi}\int_0^\pi lsin\theta d\theta$$

Computing this integral [2] gives the proportion to be $\frac{2l}{\pi}$. Thus, the probability that a randomly thrown needle of length l intersects a line in the plane ruled with parallel lines of one unit distance apart is $\frac{2l}{\pi}$. Pi can thus be approximated experimentally using this method.

METHOD

Drop many needles of a known length l (where $l < 1$) onto a plane with lines one unit apart. Find the proportion of needles that intersect a line (as the number of needles reach infinity, this proportion will be closer to the probability calculated above). Divide $2l$ by this proportion to obtain an approximation for pi.

CONCLUSION

Different fields of math are united in the calculation of these classical numbers. Although it seems that these concepts are all very specialized and limited, in actuality, they are interconnected. Pi, an irrational number, can be approximated using probability, and irrational numbers can also be calculated using probability. In an interdisciplinary world, there needs to be more flexibility within disciplines.

FOOTNOTES

[1] The area of the function $f(x)$ over the interval $[a,b]$ can be represented by the integral

$$\int_a^b f(x)\,dx$$

[2] The steps in solving the integral:

$$\frac{1}{\pi}\int_0^\pi lsin\theta d\theta =$$

$$\frac{l}{\pi}\int_0^\pi sin\theta\, d\theta \;=$$

$$\frac{l}{\pi}\int_0^\pi (-cos\theta)\, d\theta =$$

$$\frac{l}{\pi}(-cos(\pi)) - \,l\,(-cos(0)) =$$

$$\frac{2l}{\pi}$$

Unmasking Recurrence Sequences

BY ANTONELLA PERUCCA
Associate Professor in Mathematics and its Didactics, University of Luxembourg

There are many interesting sequences of numbers that can be described by the first values ('initial terms') and by a rule ('recurrence').

The Fibonacci sequence

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

starts with the numbers 0 and 1. All other terms can be obtained with this simple rule:

Add the two previous terms.

Indeed, we have 0 + 1 = 1, then 1 + 1 = 2, then 1 + 2 = 3, then 2 + 3 = 5, and so on.

The sequence is a stripe of numbers, and the recurrence is a mask!

0	1	1	2	3	5	8	13	⋯

$$\square + \square = \square$$

The recurrence relates neighbouring terms in a prescribed way: You always see a true equality when the mask goes on the stripe!

0	1	1	2 + 3 = 5	8	13	⋯

The recurrence lets you deduce all terms from the initial ones: You can use the mask to compute the next term!

0	1	1	2 + 3 = $\square$		

Periodic sequences (for example the sequence 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, . . .) regularly repeat finitely many values. The mask looks like this:

$$\square \quad = \quad \square$$

Arithmetic sequences (for example the sequence 1, 3, 5, 7, 9, 11, . . .) require you to add one same number N at each step. The mask looks like this:

$$N + \square = \square$$

Geometric sequences (for example the sequence 1, 2, 4, 8, 16, 32, . . .) require you to multiply by one same number N at each step. The mask looks like this:

$$N \times \square = \square$$

You can produce stripes and masks out of paper (or program a small animation) to try things out!

Q1: Can you find two different masks for the following sequence?

1	-1	1	-1	1	. . .

Q2: What happens to the Fibonacci sequence if you start with the values 1 and 1 instead? And what if you start with 0 and 0?

Q3: Which sequence do you get if you change the recursion of the Fibonacci sequence by turning the addition into a subtraction?

0	1					. . .

$$\square - \square = \square$$

Q4: For periodic/arithmetic/geometric sequences, what is the mask to go backwards (i.e. to compute previous terms)?

Refer to www.antonellaperucca.net for the solutions

Introducing Permutation Groups via Magic

BY RICARDO TEIXEIRA

Assistant Professor of Mathematics at the University of Houston

Introduction

World-famous magicians have developed routines whose justifications are clever applications of mathematics. However, rarely do the explanations get proper attention. Here, we propose an innovative way of relating Permutation Groups to Magic.

Some Magic Tricks

Some magic tricks involve high-level mathematical principles. Let's illustrate by detailing the explanations of a trick performed by a fictitious magician, Gisele.

A volunteer, Arthur, displays three objects in line. After performing several permutations(that is, rearranging the order of the objects), he mentally chooses an object. He realizes some other permutations. And the magician, Gisele is able to infer which the chosen object was.

The above trick involves finding a hidden piece of information: the chosen object. Its explanation involves *permutation groups*.

Several fields of mathematics and applied mathematics have been related to magic tricks. For instance, Rungratgasame et. al. [4] use magic squares as examples of vector spaces and other *linear algebra* concepts. *Number theory* may be used in card tricks: Teixeira [6] used numeral systems, and Eisemann [1] applied cyclic permutations. More group theory can be seen in relation to magic when Graham [2] connected semidirect product to card shuffling. Simonson and Holm [5]used card tricks to teach discrete mathematics.

Basic probability is illustrated by Teixeira [8]; and more advanced probability theory is taught by Lesser and Glickman [3]. Finally, computer algorithms for error detection and correction are exemplified in a trick in a paper by Teixeira [7].

Groups and Finite Groups

For completion, we use this section to summarize concepts we need in our next one.

The trick is based on an algebraic structure called a *Finite Group*. In *Abstract Algebra*, a college-level course, we study more general algebraic structures, than the usual number system. The main ones are groups, fields and rings.

Simply put, a group is a generalization of the structure "integers equipped with addition."

We extract four main properties of this addition structure:

1. Take two integers, add them, then the result is also an integer. In mathematical terms: if a and b are integers, then $a + b$ is also an integer.

2. If there are three integers to be added $a + b + c$, then we could initially group the first two numbers or the last two numbers that the result would be the same. Mathematically: if a, b and c are integers, then $(a + b) + c = a + (b + c)$.

3. Also, under addition there is a number that does not modify the other numbers when added: the number zero. It is called the "identity:" if a is an integer, then $a + 0 = a$ and $0 + a = a$.

4. Finally, for every integer, there is an inverse, that is, we could add a number to it such that the result is the "identity element," or in other words, the sum is zero. This inverse is called "additive inverse." In symbols: if a is an integer, then there is another integer, b, such that $a + b = 0$, and $b + a = 0$. The additive inverse of an integer a is represented by the symbol "a." (a multiplicative inverse may be represented by a^{-1}.)

Notice that the commutative property is not required. That is, changing the order of elements may also change the result, in some groups.

We generalize this structure by defining:

A group is a set of elements G equipped with and operation + having the following four properties:

1. Closure: if a and b are elements of G, then $a + b$ is also an element of G;

2. Associativity: if a, b and c are elements of G then $(a + b) + c = a + (b + c)$;

3. Existence of an Identity Element: there is an element e such that for all element a of G, $a + e = e + a = a$.

4. Existence of Inverse: for every element a of G, there is an element b of G such that $a + b = b + a = e$.

Some Examples

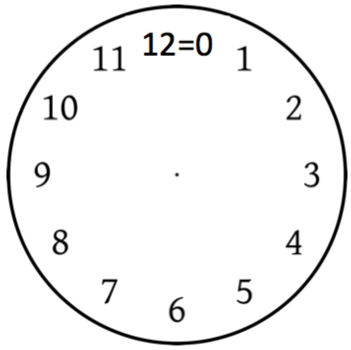
In order to better understand what a group is, let's discuss some structures composed by a set and an operation that are groups and some that do not fulfill the definition of group.

For instance, the structure *integers equipped with multiplication* does not fulfill the last property: there are integers that do not have a *multiplicative inverse*. For instance, the multiplicative inverse of 3 is 1/3 which is not an integer.

Even when we expand the previous example to the set of *all Real numbers with multiplication* still does not fulfill all properties, since there is a Real number that does not have a multiplicative inverse: zero has no inverse.

We could exclude this single number and consider *all non-zero Real numbers, equipped with multiplication* and all properties are now satisfied, with the multiplicative identity being the number 1. The group is represented by $\{\mathbf{R}^*, \times\}$ (in general, the asterisk simply means that zero is removed).

Consider the idea of finding what the time of day is after few hours. For example, on a 12-hour clock, if it is 9 o'clock, after 6 hours, what time will it be?



We do this computation a lot in our daily lives. We have an example of a finite group. The number 12 is the same as our additive 0, since 12 plus any hour is equal to the hour itself. So, what is 9 + 6? Starting at 9, after 3 hours we reach 12, or zero. So,
after 3 more hours, it will be 3 o'clock.

What is the additive inverse of 4, for instance? It is the number from the set such that when added to 4 is equal to the additive identity, or zero. In other words, it must be 8, since 4 + 8 = 12 = 0.

Mathematically, the group is the set of elements $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ equipped with the operation of addition.

We can better understand the group by checking its table of addition:

+	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	2	3	4	5	6	7	8	9	10	11
1	1	2	3	4	5	6	7	8	9	10	11	0
2	2	3	4	5	6	7	8	9	10	11	0	1
3	3	4	5	6	7	8	9	10	11	0	1	2
4	4	5	6	7	8	9	10	11	0	1	2	3
5	5	6	7	8	9	10	11	0	1	2	3	4
6	6	7	8	9	10	11	0	1	2	3	4	5
7	7	8	9	10	11	0	1	2	3	4	5	6
8	8	9	10	11	0	1	2	3	4	5	6	7
9	9	10	11	0	1	2	3	4	5	6	7	8
10	10	11	0	1	2	3	4	5	6	7	8	9
11	11	0	1	2	3	4	5	6	7	8	9	10

And the additive inverse values are:

number	0	1	2	3	4	5	6	7	8	9	10	11
inverse	0	11	10	9	8	7	6	5	4	3	2	1

So, following the convention to call the additive inverse of a number as its *negative*, it makes sense to say that -4 = 8, for example. Which in the clock case means that if the hour hand is at a certain time, then going 8 hours forward (clockwise direction) or going 4 hours backwards (counter-clockwise direction) the final position will be the same.

Another example of a very familiar finite group is the days of the week. If today is Wednesday, which day of the week will be in 16 days? In this case we are dealing with a group of 7 elements:

Sun	Mon	Tues	Weds	Thurs	Fri	Sat
0	1	2	3	4	5	6

So, if today is Wednesday, or 3, so in 16 days it will be 3 + 16 = 19 = 5, which means that it will be Friday. We notice that we need to remove multiples of 7 in this case. So 19 will be 19 – 7 – 7 = 5.

These groups are sometimes called (Z_n, +) or simply Z_n, where n is the number of elements. It is simply the remainder of the division by *n*: in Z_7, 89 = 5, since 89 = 7 x 12 + 5. In other words, if today is Monday (1), in 89 days it will be 1 + 89 = 1 + 5 = 6, Saturday.

Multiples

Consider for a moment the "clock"-group Z_{12}. Let's see which the multiples of some elements are. For example, if we want to see the multiples of 4: The first multiple is 4 x 0 = 0, then 4 x 1 = 4, then 4 x 2 = 8, but see what happens on the next multiple: 4 x 3 = 12 = 0. If we continue, 4 x 4 = 16 = 4, 4 x 5 = 20 = 8, 46 = 24 = 0. We notice that we will be stuck on the values {0, 4, 8}.

However, the multiples of 5 behave differently, they are not a mere subset of the group, they are the entire group. The multiples of 7 would also include all elements of the group.

A natural question that arises is: What if the group is Z_{13}, will any set of multiples be different from Z_{13}? Why or why not?

Permutations

If there are *n* objects lined up in a certain order, and someone rearranges them in some new order, we say that this person is creating a new *permutation* of the objects.

For instance, suppose there are five objects: *A*, *B*, *C*, *D*, and *E*. First they are lined up in alphabetical order: *ABCDE*. Someone messes up the order and create a new permutation *CDBEA*. We notice that *A* was in the first position and was sent to the fifth position. The fifth position used to be *E* which was sent to position four. Position four was letter *D* which was sent to position two. Position two was the place for the letter *B* which now has position three. Finally, position three had letter *C*, but after the rearrangement became in position one. Hence, the transformation can be summarized as:

$$1 \to 5,\ 5 \to 4,\ 4 \to 2,\ 2 \to 3,\ 3 \to 1$$

Permutations will be described inside parentheses as: start with the first position that is not kept intact, then write the position to which the object was sent. Then continue by enumerating, in order, the new positions of the objects in the previous position. When an object is sent to the initial position, simply close the parentheses. If there are less than 10 positions, we do not separate the numbers. In our example, the permutation can be simply represented by (15423).

How would we represent the transformation *ABCDE* → *DCBEA*?

Notice that 1 → 5, 5 → 4, 4 → 1, 2 → 3, 3 → 2. So apparently there are two independent permutations (154) and (23). So we represent this second transformation as (154)(23).

Similarly, the transformation *ABCDE* → *DBCEA* is (154) and the two elements did not change positions, two and three do not need to be written.

Consider any permutation described above. By reverting the arrows, we create its "inverse." For example, what is the inverse of (154)? We need 5 → 1, 4 → 5 and 1 → 4. Hence $(154)^{-1} = (145)$. Similarly, $(23)^{-1} = (23)$.

Can we combine permutations? What if you have the following instructions:

- Realize the first permutation (15423);
- Then, realize the second one (154)(23).

What would the result be?

Notice that the permutations are about the positions and not the elements. We can track each element:

- Letter A goes from position one to position five in the first permutation, then from position five to position four. So we start by putting (14 _ _ _).

- Now we track what happens to the element that was in position four, letter D. It goes from four to two, and then from two to three. So, our transformation becomes (1 4 3 _ _).

- Position three's element goes from three to one, then from one to five: (1 4 3 5 _).

- Letter E goes from five to four, and from four to one. So we close the cycle: (1 4 3 5).

- Finally, B goes from two to three, then from three to two. So it does not change its position, so we don't need to represent.

The combination of permutations acts like an operation on a group:

1. Closure: when permutations are combined, the result is just another permutation.

2. Associativity: if there are three permutations p_1, p_2, and p_3 to be realized in order (first p_1, then p_2 and then p_3), we could see the result of first combining p_1 and p_2 and then realizing p_3 and the result would be the same as combining p_2 and then p_3 and make the combination of first p_1 then the result of p_2 and p_3. (Exercise)

3. Existence of an Identity Element: there is an element (1) which fixes every position such that for all permutation, if we combine with (1) then the permutation is not affected.

4. Existence of Inverse: for every permutation, we can find its inverse by simply undoing the commands.

We follow the usual notation, where the first permutation is on the right, and then we go left. For instance, the description above can be summarized as:

$$[(154)(23)](15423) = (1435)$$

In the trick mentioned earlier, objects simply change positions within themselves, that is, the volunteer is performing permutations.

Mentalism: Guessing an Object

For this trick, three different objects will be needed. They could be, for instance, paper ball, pen, eraser, ring, earring, key, die, a card, etc.

The magician lays the objects on a flat surface and name each position from left to right as 1, 2, and 3. For simplicity, use the volunteer's point of view.

Suppose, for example, that the objects are an arrow, a ball, and a cookie, and that they are displayed in this order: ABC.

1. According to the volunteer's (Arthur) point of view, Gisele lays from left to right: Arrow, Ball, and Cookie.

2. She explains that Arthur can realize permutations of only two objects at a time.

3. Gisele asks Arthur to mentally select an object, this will be used later.

4. The magician turns her back to the objects.

5. Arthur is free to make as many permutations as he wants. He simply says loudly the positions he is exchanging, not the objects. For example, "two and three."

6. Once he is satisfied with the permutation in front of him, he lets Gisele know. At that point, Gisele gives a special instruction:
   - Now, do not tell me the positions you are switching, remember your chosen object and simply switch the positions of the other two objects.

7. Then Arthur can make more permutations, and, as before, he says the positions he is exchanging.

8. Once Arthur is satisfied with the result, he asks Gisele to turn back, and Gisele can say which object Arthur chose.

Trick: Gisele knows the initial configuration, so she knows, for instance, the position that the arrow is in the beginning. She will be mentally tracking this object as if that special move (step 6) never happens. There are two possibilities:

- The arrow is indeed in the position that she tracked: the chosen object was the arrow.

- The position in which she expected the arrow to be is not occupied by the arrow, then the chosen object is not the arrow, nor the object on that position. It is the remaining one.

Explanation: If the arrow is in the position she tracked, then that means that the special move did not affect the arrow, only the ball and the cookie. That means that the arrow was the chosen one.

If, say, the cookie is where the arrow was expected to be, that means that at that special move, arrow and cookie switched places. Which means that the ball was the chosen object.

Example: For example, suppose that Arthur says in order "1 and 2", "2 and 3", "1 and 2", "2 and 3". The result is (23)(12)(23)(12) = (123). But Gisele only needs to know that $1 \rightarrow 2$. That is, Gisele knows that the arrow is on position two.

When the special move comes, Arthur realizes either (12) if the chosen object is on position three (if the ball is the chosen object), (23) if the chosen object is on position one (the cookie), or (13) (the chosen object is on position two (arrow). The results could be either:

- (12)(123) = (23) (arrow changed places with the object in position three, the chosen object is in position one). The arrow is not the chosen object, and Gisele is now tracking another not-chosen object;

- (23)(123) = (13) (arrow did not change position). Gisele continues to track the arrow, and the arrow is the chosen object;

- (13)(123) = (12) (arrow changed position with the object in position one, the chosen object is in position three), The arrow is not the chosen object, and Gisele is now tracking another not-chosen object.

Differently from some other tricks, this one can be repeated several times, with different objects, if desired.

Final Remarks

With this work, a possibility of presenting permutation group via a magic trick is introduced. We were also able to reveal magic secrets in a rigorous mathematical language.

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Ricardo Teixeira is an Assistant Professor of Mathematics at the University of Houston - Victoria. Dr. Teixeira has created innovative ways to use high-level mathematics to explain magic tricks. To this date, his live "Mathemagics" performances have been seen by over 3,000 students.

WHY IS $\sqrt{2}$ IRRATIONAL?

BY SUSAN E. HODGE AND ADAM H. GREENBERG.

Susan Hodge: Department of Pediatrics, The Ohio State University, Columbus, Ohio

Adam Greenberg: Raytheon Corporation, El Segundo, California

Introduction

This is a story of how we came to truly understand the irrationality of $\sqrt{2}$. The story stretches over several decades, and involves detours into different number bases. We believe that the proof we finally reached is both more fundamental and more intuitive than the proof usually taught in school [1]. We hope readers will gain some new insights from this account, as well as enjoying our meandering journey to reach our conclusion.

Standard Proof (Proof 1)

The story begins when the first author was in 7^{th} grade and learned what we call Proof 1:

Assume $\sqrt{2}$ can be written as a fraction $\frac{p}{q}$ reduced to lowest terms, where p and q are integers. Since $\sqrt{2} = \frac{p}{q}$, it follows that $\left(\frac{p}{q}\right)^2$=2, and therefore

$$p^2 = 2q^2 \qquad \textbf{(1)}$$

From (1), p^2 is even, which means that p itself is even (since odd times odd equals odd). So p can be written as $2n$, where n is an integer, and thus, $p^2 = 4n^2$; that is, p^2 not only is an even number but is actually divisible by 4. Rewrite (1) as $4n^2 = 2q^2$, implying $q^2 = 2n$, i.e., q^2 is even as well, and thus so is q. Thus p and q are both even, which violates the opening assumption that $\frac{p}{q}$ had been reduced to lowest terms. This is a contradiction; therefore $\sqrt{2}$ cannot be written as a fraction $\frac{p}{q}$. *QED*

The first author found this standard proof singularly unsatisfying. It proves that you can't reduce $\sqrt{2}$ to "lowest terms," but it doesn't really explain why $\sqrt{2}$ can't be written as a rational number. It left her wondering, "What if we *didn't* assume $\frac{p}{q}$ had been reduced to its lowest terms? What does that have to do with the square root of 2?"

Proof by Rightmost Digit (Proof 2)

Many years later, the second author was in 7th grade and had a similar reaction. He thought he had a simpler, more obvious proof: "We know $\sqrt{2}$ is between 1 and 2, so expressed in decimal form it will be written as 1 point something. Consider a decimal such as, e.g., 1.4. Since its rightmost digit is 4, when squared it must end with a 6 in the rightmost position after the decimal point. I.e., it cannot have all zeroes after the decimal point, and thus cannot equal exactly 2. Similarly for all other nonzero rightmost digits (4 and 6 squared yield 6; 1 and 9 squared yield 1, etc.). So any decimal number between 1 and 2, when squared, cannot end up with all zeroes to the right of the decimal point and therefore cannot equal 2.0." This was Proof 2, Proof by Rightmost Digit.

The first author was struck by this insight but pointed out that the rightmost-digit reasoning can be applied only to fractions that "terminate" when in decimal form (e.g. $\frac{3}{5}$ = .6), not to "repeating" decimals, such as $\frac{14}{99}$ = .1414 … since they do not actually have a rightmost digit. (A repeating decimal is one whose digits are periodic, with the infinitely repeated portion not equaling zero.) Even if the repeating figure consists of only one digit, e.g., $\frac{1}{3}$ = .333 …, there is still no rightmost digit. (And of course, irrational numbers also have no rightmost digit.) Thus, the Proof by Rightmost Digit works for terminating decimals but not for repeating ones.

Detour: Terminating vs. Repeating Decimals

Several years went by. The first author kept mulling over this on the back burner. What is the difference between terminating and repeating decimals, anyway? She thought about unit fractions (fractions of the form $\frac{1}{n}$ where n is an integer). Some unit fractions, such as 1/16 and 1/200, convert into terminating decimals (.0625, .005, etc.), whereas others, such as 1/7 and 1/11, have decimal equivalents that go on forever. Terminating decimals are easier to write, and also easier to work with (for example, to multiply). We call an integer n "friendly" if its unit fraction $\frac{1}{n}$ is terminating, and "unfriendly" if $\frac{1}{n}$ is repeating.

What determines whether a given integer is friendly or unfriendly? The answer is that $n = 2^p5^q$ (with p and q both nonnegative integers, at least one > 0) is a necessary and sufficient condition for n to be friendly. Why those two integers, 2 and 5, as opposed to, say, 2 and 3? It's because they are factors of ten, the base we write our numbers in. If we wrote our numbers in base twelve, 2 and 3 would be friendly, and 5 would be unfriendly. Or consider 7, which is unfriendly in base ten ($\frac{1}{7}$ = .142857142857...). But in base seven, seven is written as 10, and one-seventh is written as 1/10, for which the decimal form is 0.1 (terminating). In fact, any integer q is friendly when written in base q, where it will appear as 0.1.

In other words, whether a given fraction is terminating or repeating is not inherent, but rather depends only on what base we choose to write in. After all, the numbers themselves have an absolute meaning, but the base system is just a "bookkeeping" method. The number we call "twelve" may be written "12" (base ten), "15" (base seven), or "C" (base sixteen), but whichever way we write it, it always represents this many: ||||| ||||| ||.

Proof by Rightmost Digit Revised (Proof 3)

End of detour and back to our story. As we saw, Proof 2 was valid for terminating decimals but could not be applied to numbers in repeating decimal form. However, we now knew, from the detour, that any fraction can be expressed as a terminating decimal, if we choose an appropriate base to write it in. Specifically, for any given rational fraction between 1 and 2, say $T = \frac{p}{q}$, which is a candidate to equal $\sqrt{2}$, it is possible to find a base system in which T can be expressed in terminating decimal form, and such that T^2 has at least one nonzero digit to the right of the decimal point. (This statement can be justified rigorously; details not included here.) Therefore, since T^2 cannot equal an integer; T cannot equal $\sqrt{2}$; and 2 does not have a rational square root.

And since the base represents only a bookkeeping method, it suffices to find *one* base in which our candidate T does not yield 2 when squared. If T^2 doesn't equal 2 in base b, it doesn't equal 2 in any base.

Back to Fractions (Proof 4)

With Proof 3, we had accomplished our goal of creating a more satisfying proof of the irrationality of $\sqrt{2}$ than the one we had learned in 7th grade – a proof immediately showing exactly *why* a rational number between 1 and 2, when squared, cannot equal an integer. Moreover, in the process, we had learned things about terminating and repeating decimals and about the role of different base systems.

However, Proof 3 is not elegant, and its relative clunkiness lurked in the backs of our minds. The "rightmost digit" argument was a useful tool, a means to reach an end. But the more fundamental insight is that if a rational number has any nonzero digits to the right of the decimal point, then the square of that number must also have at least one nonzero digit to the right of the decimal point. Or, even more simply: *If a rational number is not an integer, then its square cannot be an integer, either*.

It turns out that it is much more elegant to show this result by *returning to fractions*, written as one integer over another integer, rather than numbers in decimal form. But first, an exercise:

> Let us try to create a fraction that, when squared, equals 2. "How hard could it be?" First attempt is 7/5. Does that equal $\sqrt{2}$? No, because $(7/5)^2 = 49/25 = 1.96$ – a little too small. Increment that by a couple of hundredths: $(142/100)^2 = 2.0164$ – a little too large. Continue like this, e.g., $(1{,}414/1{,}000)^2 = 1.999396$ and $(1{,}415/1{,}000)^2 = 2.002225$, but we will always under- or over-shoot. This exercise can help us understand why the ancient Greeks were puzzled and frustrated by the concept of irrationality.

Why can we not eventually converge to a fraction that, when squared, will equal 2? Look again at the first guess, $\frac{7}{5}$. It is not an integer, because 5 does not go evenly into 7. When we square $\frac{7}{5}$ we get $\frac{7\times7}{5\times5}$. Since 5 does not go evenly into 7, 5 x 5 also does not go evenly into 7 x 7, and therefore (7 x 7)/(5 x 5) is also nonintegral. This is why squaring a rational noninteger yields another rational noninteger. Or in simpler terms: A nonintegral fraction times itself equals another nonintegral fraction.

More formally: Consider any nonintegral fraction [2], with the numerator and denominator each broken down into its unique prime factorization, e.g., $T = \frac{p_1^{n_1}p_2^{n_2}\cdots p_i^{n_i}}{q_1^{m_1}q_2^{m_2}\cdots q_j^{m_j}}$. (It does not matter if some of the primes in the numerator equal some of those in the denominator; i.e., the fraction need not be reduced to lowest terms; cf. Proof 1.) The denominator contains at least one prime factor (call it s) at a higher power than in the numerator. Say the numerator of T contains s^n, and the denominator, s^m, with $0 \le n < m$. The squared fraction has s^{2n} in the numerator and s^{2m} in the denominator. Since $2n < 2m$, this squared fraction is also nonintegral. (Note that unique factorization is critical for this argument also, in order to ensure that squaring T does not introduce *additional s* factors to the numerator, such that the numerator would contain s^k, with $k \ge 2m$.) [3] This argument proves:

Lemma. Any nonintegral fraction, when squared, equals another nonintegral fraction.

That is, an integer squared equals an integer, whereas a nonintegral fraction squared equals a nonintegral fraction. (Among rational numbers, the set of integers and the set of nonintegral fractions are each closed

under squaring.)
Thus there are only two possibilities for square roots of integers n: Either n is a "perfect square" (1, 4, 9, etc.), in which case its square root is an integer, or n is not a perfect square, in which case its square root, which falls between two integers, cannot be a fraction (by the Lemma) and therefore is not rational. Since 2 is not a perfect square, its square root is not rational. *QED*.

Concluding Remarks

To summarize, we went through four stages. We were dissatisfied with the standard proof (Proof 1) taught to us in 7th grade, which depended on a clever trick but did not convey the essence of why the square root of 2 is not rational (cannot be written as a fraction). Our first insight concerned the "rightmost digit" (Proof 2). This insight was good as far as it went, but applied only to terminating fractions, not repeating ones. We solved that problem by working with different base number systems, along with the realization that the base system is only a bookkeeping device and does not affect the actual meaning of the number (Proof 3). And finally we came to the most fundamental insight, which was actually the foundation for Proof 2, although we did not realize it originally – namely, that if a number is a nonintegral fraction, then its square is also a nonintegral fraction. Therefore, for any integer that is not a perfect square, its square root is irrational (Proof 4).

As a final exercise, we revisit Proof 1. Start the same as before, i.e., assume that $\frac{p}{q}$ is the square root of 2, so equation (1) holds. But then instead of stipulating that p/q is reduced to lowest terms, apply the insight of Proof 4: Since $\frac{p}{q}$ is not an integer (since it falls between 1 and 2), there must exist at least one prime factor (call it s), occurring at higher power (n) in q than its power (m) in p. This means that the right side of (1) includes s^{2n}, as opposed to s^{2m} on the LHS. It is impossible for an equation to have a prime factor at a higher power on one side than on the other. Therefore, $\frac{p}{q}$ cannot be the square root of 2. *QED*. There is no need to reduce $\frac{p}{q}$ to lowest terms or to go through the convoluted steps in Proof 1.

In this piece we try to offer some insight into ways of thinking about fractions and, in particular, an appreciation of the fundamental fact that, if the denominator does not go evenly into the numerator, then the denominator squared does not go evenly into the numerator squared. That is the crux of why $\sqrt{2}$ is irrational.

Footnotes

[1] Our final proof is not original to us, although we did not know that at the time. See, e.g., Wikipedia, "Square root of 2 [https://en.wikipedia.org/wiki/ Square_root_of_2#Proofs_of_irrationality]."

[2] A fraction is nonintegral if and only if the denominator contains at least one prime factor at higher power than in the numerator (zero is also a power). Note that this definition depends critically on the unique prime factorization of the Fundamental Theorem of Arithmetic. For if, say, 6 had two distinct prime factorizations, e.g., $6 = p_1p_2$ and also $6 = q_1q_2$, with p_1, p_2, q_1, and q_2 all distinct and prime, then the fraction $3p_1p_2/(q_1q_2)$ would satisfy the above condition but would not be nonintegral.

[3] Given a composite number $A = P_1^{n_1} \cdots P_i^{n_i}$, in standard prime factorization. Consider A^2, which can be found by squaring each component of the prime factorization of A, i.e., $A^2 = P_1^{2n_1} \cdots P_i^{2n_i}$. By unique factorization, an alternative prime factorization – one that would contain another prime, say, q – does not exist. This proves that squaring a composite number cannot introduce any new primes.

2019 MATH QUICKIES

Solutions will be published in the next issue of PI IN THE SKY

1. **You start from the triple $(a_1, a_2, a_3) = (2, 3, 10)$, and can perform the following operations: choose two numbers a_i and a_j, $i \neq j$ and a number θ; you then replace them by $a_i\cos\theta + a_j\sin\theta$ and $a_i\sin\theta - a_j\cos\theta$. Is it possible to obtain the triple (5, 9, 2) by repeatedly applying operations of this type (where each time, you can choose which i and j to modify, and you can choose the number θ)?**

2. **Show that the equation $15x^2 + y^2 = 4n$ has positive integer solutions for any integer $n \geq 2$.**

3. **Is it possible to partition a square into a number of congruent right triangles having an angle of 30°? Justify your answer.**

4. **Let A be a set of n integers. Prove that there exists a non-empty subset of A such that the sum of its elements is divisible by n.**

5. **Let $P(x)$ be a polynomial with integer coefficients, such that $P(2018) \bullet P(2019) = 2021$. Show that there is no $k \in \mathbb{Z}$ such that $P(k) = 2020$.**

6. **Let $a_0, a_1, \cdots, a_n, \cdots$ be a sequence of real numbers such that $a_{n+1} \geq \dfrac{a_n^2 + 3}{4}$ $n \geq 0$. Prove that $\sqrt{a_{n+4}} \geq \dfrac{a_{n-4}^2}{8}$, for any $n \geq 4$.**

7. **The convex quadrilateral $ABCD$ is inscribed in a circle of centre O and its diagonals intersect at E. The projections of E on AB, BC, CD, and DA are the points M, N, P, and respectively Q. Prove that the area of $MNPQ$ is half the area of $ABCD$ if and only if $ABCD$ is a rectangle.**

PRIZE!

PIMS is sponsoring a prize of \$100 to the first high school student (from within the PIMS operating region: Alberta; British Columbia; Manitoba; Saskatchewan; Oregon; Washington) who submits the larges number of correct answers before December 1, 2019. Submit your answers to: pims@uvic.ca

2017 MATH QUICKIES

1. Find the number of the nonempty subsets of the set of the first nine positive integers which contain the same number of even and odd integers.

   Solution:
   There are five odd integers and four even integers. There are $\binom{5}{k}$ ways to chose of k integers from 5 and $\binom{4}{k}$ ways to choose k integers from 4. The number of the requested subsets is $\binom{5}{1}\cdot\binom{4}{1}+\binom{5}{2}\cdot\binom{4}{2}+\binom{5}{3}\cdot\binom{4}{3}+\binom{5}{4}\cdot\binom{4}{4}=20+60+60+5=125$.

2. Find the tens digit of $2^{2016}+2^{2017}$.

   Solution:
   The given number can be written as following:

$$2^{2016}+2^{2017}=2^{2016}(1+2)=3\cdot 2^{2016}=3\cdot 2^{16}\cdot 2^{2000}=3\cdot 2^{16}\cdot (2^{20})^{100}.$$

   The last two digits of $2^{16}=2^8\cdot 2^8$ are 36 and the last to digits of $2^{20}=(2^{10})^2=(1024)^2$ are 76. If a number ends in 76 then any natural power of it ends in 76 and hence $(2^{20})^{100}$ ends in 76. Since the last two digits of $3\cdot 2^{16}$ are 08 and the last two digits of $(2^{20})^{100}$ are 76 we find that the last two digits of the given number are 08.

3. If $a_{n+1}=a_n+3a_{n-1}, n\geq 2$ and $a_1=a_2=1$, find the remainder when a_{2016} is divided by 5.

   Solution:
   Evaluating $a_n(\text{mod } 5)$ for $n\geq 2$ one obtains: $a_3\equiv 1+3\cdot 1\equiv 4 \pmod 5$, $a_4\equiv 4+3\cdot 1\equiv 2 \pmod 5$, $a_5\equiv 2+3\cdot 4\equiv 4 \pmod 5$, and so on. Hence the sequence $a_n(\text{mod } 5)$ is

$$1,1,4,2,4,0,2,2,3,4,3,0,4,4,1,3,1,0,3,3,2,1,2,0,1,1,4,2,\cdots$$

   and its terms repeats after 24 steps. Since $2016\equiv 0 \pmod{24}$, hence $a_{2016}\equiv a_0\equiv 1 \pmod 5$.

4. Let x,y,z be any real numbers such that $3x+y+2z\geq 3$and $-x+2y+4z\geq 5$. Find the minimum value of $7x+5y+10z$.

   Solution:
   Let A,B be real positive numbers. Then

$$A(3x+y+2z)+B(2y-x+4z)\geq 3A+5B \iff (3A-B)x+(A+2B)y+(2A+4B)z\geq 3A+5B.$$

   If $3A-B=7, A+2B=5, 2A+4B=10$ that is equivalent to $A=\frac{19}{7}, B=\frac{8}{7}$ the above inequality can be written as

$$7x+5y+10z\geq \frac{97}{7}.$$

   Hence the minimum value of $7x+5y+10z$ is $\frac{97}{7}$.

5. Find all real numbers x which are solutions of the equation $\sqrt[3]{x^2+x+\sqrt[3]{x^2+x+2}}=2$.

   Solution:

   Let $\sqrt[3]{x^2+x+2}=y$. Then $x^2+x=y^3-2$ and the given equation can be written as $\sqrt[3]{y^3+y-2}=2$ or equivalently $y^3+y-10=0$ that is, $(y-2)(y^2+4y+7)=0$. The only real solution is $y=2$, hence $x^2+x-6=0$ and thus $x\in\{-3,2\}$.

6. Find the minimum value of $\frac{c}{a+b}+\frac{a}{b+c}$, were $a,b,c \in [2,3]$ and $a+b+c=8$.

   Solution:

$$\frac{c}{a+b}+\frac{a}{b+c}=\frac{8-(8-c)}{8-c}+\frac{8-(8-a)}{8-a}=8\left(\frac{1}{8-a}+\frac{1}{8-c}\right)-2$$

   The function $F(a,b)=\frac{1}{8-a}+\frac{1}{8-c}$ that is symmetric in a and c should attain the minimum value when $a=c$, with a having the smallest possible value. Since $b=8-(a+c)\in[2,3]$, then $a+c\in[5,6]$ and thus $a=c=\frac{5}{2}$. Hence $b=3$ and the requested minimum is $\frac{10}{11}$.

7. Find the maximum area of a convex quadrilateral having the length of sides 2, 5, 10, 11.

   Solution:

   Let $ABCD$ be a given convex quadrilateral. If A' is the symmetrical of A with respect to the line bisector of the segment DB than area($ABCD$)=area($A'BCD$), and one of these two quadrilaterals has the sides of length 2 and 11 neighbors. We may assume that the quadrilateral $ABCD$ has sides of length 2 and 11 neighbors, for instance $|AB|=2$ and $|BC|=11$.

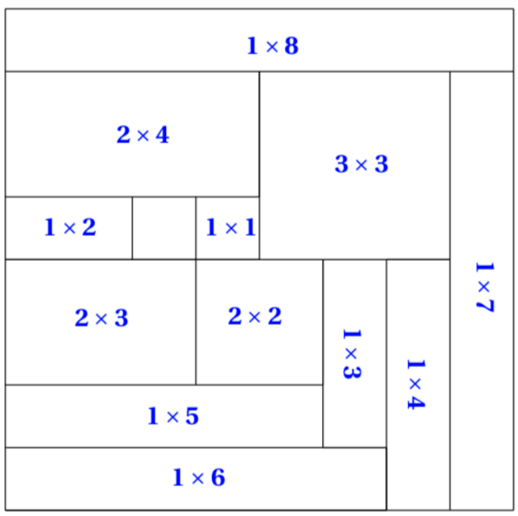
$$\text{area}(ABCD)=\frac{1}{2}(|AB||BC|\sin B+|AD||DC|\sin D)\leq\frac{1}{2}(22+50)=36.$$

   The upper limit 36 is achieved. Indeed, since $|AB|^2+|BC|^2=|AD|^2+|DC|^2=125$, we can choose B and D such that $\angle B=\angle C=90^\circ$ and the area of the obtained quadrilateral is precisely 36.

8. Find the maximum number of non-congruent integer-side rectangles, which can be obtained when an 8×8 square is cut into pieces with all cuts parallel to its sides.

   Solution:

   It is possible to cut only six types of non-congruent rectangles with area 1, 2, 3, 5, 7, and 9 (the rectangles with sides $1\times 1, 1\times 2, 1\times 3, 1\times 5, 1\times 7, 3\times 3$) and only six with area 4, 6, 8 (the rectangles with sides $1\times 4, 2\times 2, 1\times 6, 2\times 3, 1\times 8, 2\times 4$) as showed tin the picture bellow:



   Assume that is possible to obtain more then 12 noncongruent rectangles of areas $A_1<A_2<\cdots<A_{13}<\cdots$. Since $A_{13}>10$ we must have:

$$A_1+A_2+\cdots A_{12}+A_{13}+\cdots>1+2+3+5+7+9+2\cdot(4+6+8)+10=63+10=73$$

   that is a contradiction. Therefore the maximum number of rectangles with the requested properties is 12.

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