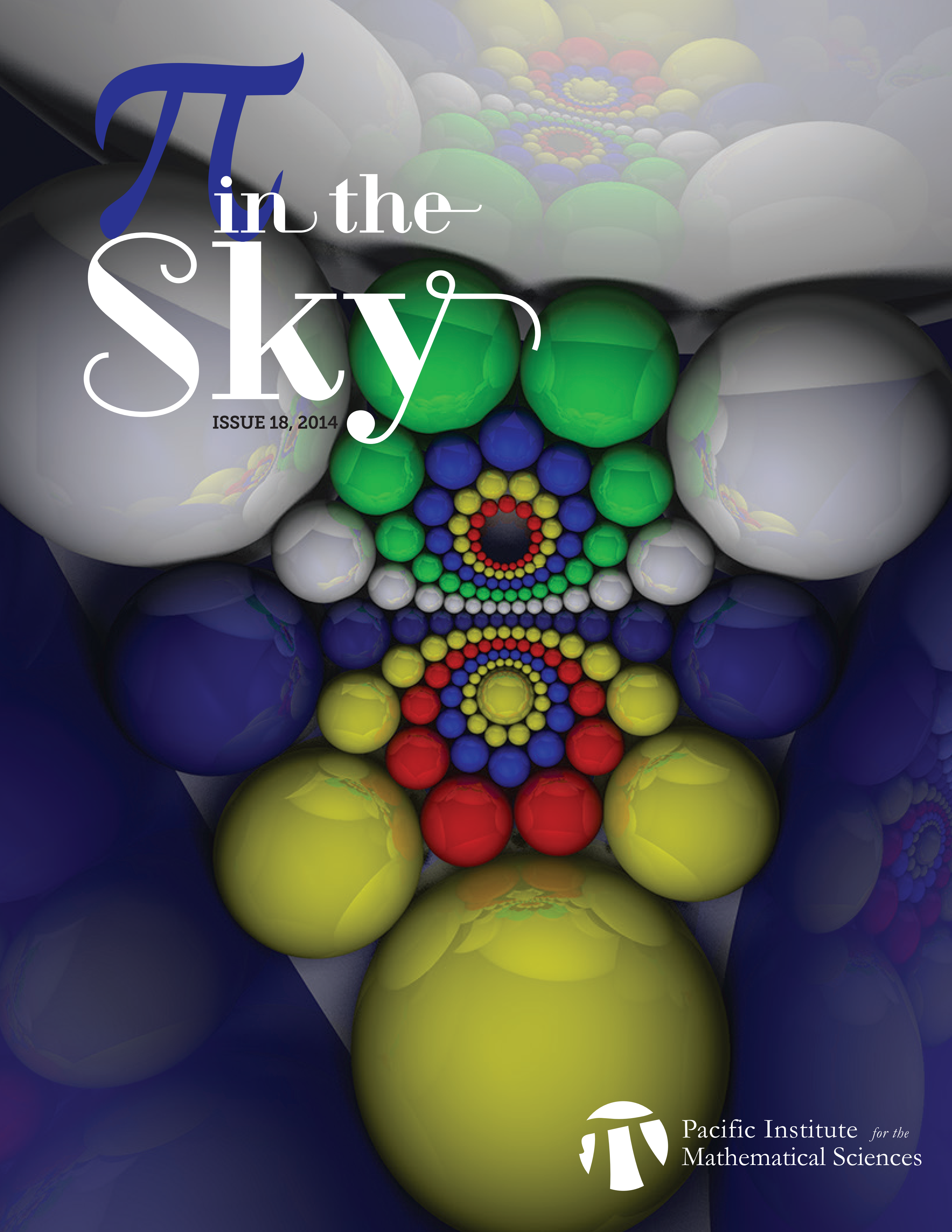
π in the Sky

ISSUE 18, 2014



Pacific Institute *for the* Mathematical Sciences

Pi in the Sky

Issue 18, 2014

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Contact Information

Pi in the Sky
PIMS University of Victoria Site Office,
SSM Building Room A418b
PO Box 3060 STN CSC, 3800 Finnerty Road,
Victoria, BC, V8W 3RT

(250) 472-4271
F/(250) 721-8958
pims@math.uvic.ca

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Pi in the Sky is aimed primarily at high school students and teachers, with the main goal of providing a cultural context/landscape for mathematics. It has a natural extension to junior high school students and undergraduates, and articles may also put curriculum topics in a different perspective.

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Melania Alvarez, UBC, Vancouver, BC
melania@pims.math.ca

Malgorzata Dubiel, SFU, Burnaby, BC
dubiel@cs.sfu.ca

Indy Lagu, U of Calgary, AB
ILagu@mtroyal.ca

David Leeming, UVic, Victoria, BC
leemingd@uvic.ca

Harley Weston, U of Regina, SK
weston@math.uregina.ca

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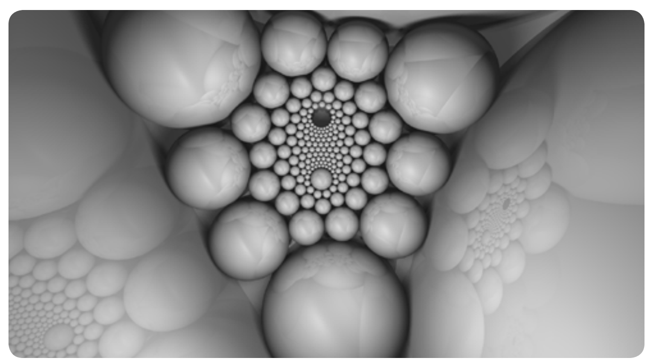
Solutions to Math Challenges at the end of this issue will be published Pi in the Sky Issue 19. See details on page 26 for your chance to win $100!

Editor's Note

As we compile this issue of *Pi in the Sky*, we have some very big news for Canadian mathematics. The most prestigious prize in mathematics (mathematics' closest equivalent to the Nobel Prize) is the Fields Medal, named after the Canadian mathematician, John C. Fields, who initiated the prize. Fields left money in his will to endow the prize and it was first awarded in 1936, four years after his death. Since then, Fields Medals have been awarded every four years to between two and four mathematicians under the age of 40.

Until this year, there had never been a Canadian Fields Medallist. That all changed when Manjul Bhargava (now 40 years old, but crucially 39 years old on January 1st of this year, which was the cut-off date!) became the first Canadian Fields Medallist at the International Congress of Mathematicians, held in Seoul, South Korea in August 2014. We have a special invited article by Andrew Granville, a Professor of Mathematics at the University of Montreal, in which he describes what Bhargava's work is all about.



The cover photo was produced by **Francesco de Comité**, a Professor of Computer Science at the University of Lille, France. The image was produced using a technique called *ray tracing*. To do this, a computer is given information about a collection of objects: their position, shape and optical properties (do they reflect a lot? a little? what colour are they?); the locations of some light sources; and the location of an observer. The computer program then traces a large number of rays from the observer's 'eye' following them as they reflect to get a computed version of what the observer would see. A typical ray-tracing image takes up to 24 hours of computer time to generate. The cover image was produced using a free computer language called *Povray*.

COMPOSING QUADRATIC FORMS: GAUSS, DIRICHLET & BHARGAVA

ANDREW GRANVILLE

1 INTRODUCTION

One of the basic problems of mathematics is to find all of the solutions of a given equation. Some equations are easy to solve, like $x + y = z$, and others can be difficult, like $x^{101} + y^{101} = z^{101}$. The level of difficulty depends largely on what kind of solutions one is looking for. If we are simply asked for complex numbers, one can take any x and y, and let z be any of the 101^{st} roots of $x^{101} + y^{101}$. If we are asked for integer solutions, then this is a problem of much greater subtlety and depth, and is one of the central problems of number theory.

One particular, but important, problem is the question of which integers are represented by a given polynomial. For example, what integers are represented by $x + 2y$, as x and y run through all the integers? Or by $u^2 + v^2$ as u and v run through all of the integers? Or even by $x^{101} + y^{101} - z^{101}$ as x, y and z run through all the integers?

In this article we will focus on quadratic equations, that is, those equations of degree two. We know a lot about this question, but by no means all that we would like to understand; there are many unsolved problems about representations by quadratic equations, some quite evidently of sublime interest. Our objective is to explain the importance of a beautiful new construction by Manjul Bhargava, allowing a new perspective on which integers are indeed represented by binary quadratic forms. But before we proceed with quadratic equations, we will review representation by linear equations and particularly, those aspects that will be relevant to our discussion of quadratic equations.

2 REPRESENTATION BY LINEAR EQUATIONS IN INTEGERS

Fix two non-zero integers a and b. We wish to determine which integers n can be represented as

$$n = ax + by$$

where x and y are also integers.

One way to start is to subtract by from both sides and divide by a so that if we are given an integer y then

$$x = \frac{n - by}{a}.$$

However, we have no guarantee that the quantity on the right-hand side will be an integer. Not only for a specific value of y, but perhaps for any value of y. So, we have not made any headway on whether there are any solutions. It is not difficult to construct examples for which there are no solutions in integers:

$$2x + 4y = 1$$

can have no solutions since, no matter what the choice of integers x and y, the left hand side will always be *even*, while the right hand side will always be *odd*. The left hand side is always even because 2 divides both terms on the left hand side (both 2 and 4), the coefficients a and b. In fact, 2 is the *greatest common divisor* (gcd) of 2 and 4, written $2 = \gcd(2, 4)$.

One can generalize this argument: If there are solutions in integers x, y to $ax + by = n$ then $\gcd(a, b)$ divides both a and b, hence both ax and by, and therefore must divide $ax + by = n$. Since $g = \gcd(a, b)$ divides a, b and n, we can divide this common factor out from each. To do this we write $a = Ag$, $b = Bg$ and $n = Ng$ for some integers A, B and N. Then,

$$Ng = n = ax + by = Agx + Bgy = (Ax + By)g,$$

and dividing by g, leaves us with

$$N = Ax + By.$$

So we have proved that if n is represented by $ax + by$ then $n = Ng$ for some integer N, and N can be represented by $Ax + By$.

For example if n can be represented by $2x + 4y$ then $n = 2N$ for some integer N, that is, n is even, and N can be represented by $x + 2y$.

Evidently, every integer N can be represented by $x + 2y$ (simply take $x = N$ and $y = 0$), so every even integer $N = 2n$ can be represented by $2x + 4y$.

This is not the whole story, for if we ask what is represented by $2x - 3y$ we cannot easily find the representation, as neither coefficient is 1, yet their gcd is 1. The trick here is to find a representation of 1, for example $2 \cdot 2 - 3 \cdot 1$ and then $n = 2 \cdot (2n) - 3 \cdot n$; meaning, we multiply the representation of 1 through by n. Therefore, for a more general linear form $ax + by$, our question boils down to finding a representation of 1. This is supplied by the *Euclidean algorithm*. We will not explain this in detail here, but its consequence is that if we are given integers a and b with gcd 1, then the Euclidean algorithm supplies us with integers u and v for which

$$au + bv = 1;$$

and therefore

$$n = ax + by \text{ where } x = nu \text{ and } y = nv.$$

3 REPRESENTATION BY QUADRATIC EQUATIONS IN INTEGERS

Let a, b and c be given integers. The polynomial

$$f(x, y) := ax^2 + bxy + cy^2$$

is a *binary quadratic* form ("binary" as in **two** variable, and "quadratic" as in degree **two**. The degree of bxy is also two, since the degree of a term like this is given by the degree of x, plus the degree of y). We are interested in what integers can be represented by a given binary quadratic form f. As for linear equations, we can immediately reduce our considerations to the case that gcd(a, b, c) = 1.

The first important result of this type was given by Fermat near the beginning of the Renaissance. He considered the particular example $f(x, y) = x^2 + y^2$, asking which integers can be written as the sum of two squares of integers. He proved two things. Firstly, that an odd prime p can be written as the sum of two squares, if and only if, $p \equiv 1 \pmod 4$ (so that 5, 13, 17, 29, 37, 41,… can be written as the sum of two squares of integers, whereas 3, 7, 11, 19, 23, 31, 43, 47,… cannot).

Secondly, that the product of two integers that can be written as the sum of two squares, can also be written as the sum of two squares, a consequence of the identity

$$(u^2 + v^2)(r^2 + s^2) = (ur + vs)^2 + (us - vr)^2; \qquad (1)$$

that is, $x^2 + y^2$ where $x = ur + vs$ and $y = us - vr$. One can combine these two facts to classify exactly which integers are represented by the binary quadratic form $x^2 + y^2$.

At first sight it looks like it might be difficult to work with the example $f(x, y) = x^2 + 20xy + 101y^2$. However, this can be rewritten as $(x+10y)^2 + y^2$, and so represents exactly the same integers as $g(x, y) = x^2 + y^2$. To see this we remark that if

$$n = f(u, v) \text{ then } n = g(u + 10v, v)$$

and if

$$n = g(r, s) \text{ then } n = f(r - 10s, s).$$

Thus, every representation of n by f corresponds to one by g, and vice-versa. This is known as a *1-to-1 correspondence*. It is obtained using the *linear transformation* $u, v \to u + 10v, v$, which is *invertible* via the inverse linear transformation $r, s \to r - 10s, s$. Such a pair of quadratic forms, f and g, are said to be *equivalent* and we have just seen how equivalent binary quadratic forms represent exactly the same integers.

It would take a whole book to fully describe the theory of binary quadratic forms. Our objective here is to study generalizations of the identity (1).

4 COMPOSITION AND GAUSS

In (1) we see that the product of two integers represented by the binary quadratic form $x^2 + y^2$ is also an integer represented by that binary quadratic form; we are now looking for further such identities.

One easy generalization is given by

$$(u^2 + dv^2)(r^2 + ds^2) = x^2 + dy^2 \text{ where } x = ur + dvs \text{ and } y = us - vr. \qquad (2)$$

Therefore, the product of two integers represented by the binary quadratic form x^2+dy^2 is also an integer represented by that binary quadratic from x^2+acy^2. For general diagonal binary quadratic forms (that is, having no "cross-term" bxy) we have

$$(au^2 + cv^2)(ar^2 + cs^2) = x^2 + acy^2 \text{ where } x = aur + cvs \text{ and } y = us - vr. \quad (3)$$

Notice here that the quadratic form on the right hand side is different from those on the left; meaning that the product of two integers represented by the binary quadratic form $ax^2 + cy^2$ is an integer represented by the binary quadratic form $x^2 + acy^2$, but not necessarily by the binary quadratic form ax^2+cy^2. For example both 2 and 3 are represented by $2x^2+3y^2$ but $2 \cdot 3=6$ is not.

One can come up with a similar identity no matter what the quadratic form, though one proceeds slightly differently depending on whether the coefficient b is odd or even. In the even case we have (with $b = 2B$)

$$(au^2 + 2Buv + cv^2)(ar^2 + 2Brs + cs^2) = x^2 + (ac - B^2)\,y^2 \quad (4)$$

where $x = aur + B(vr + us) + cvs$ and $y = us - vr$. (5)

What is the connection between the quadratic form on the left and that on the right? The most important thing to notice is that their discriminants are the same (the *discriminant* of $ax^2 + bxy + cy^2$ is $b^2 - 4ac$ and one can show that equivalent binary quadratic forms have the same discriminant). Notice that the discriminant of the quadratic form on the right side of (4), $-4(ac - B^2)$, is the same as that on the left side, $4B^2-4ac$.

What about two different binary quadratic forms; can one multiply together their values? For example,

$$(4u^2 + 3uv + 5v^2)(3r^2 + rs + 6s^2) = 2x^2 + xy + 9y^2$$

by taking $x = ur - 3us - 2vr - 3vs$ and $y = ur + us + vr - vs$. These are three different (that is, inequivalent) binary quadratic forms of discriminant -71.

Gauss called this *composition*, that is, finding, for given binary quadratic forms f and g of the same discriminant, a third binary quadratic form h of the same discriminant for which

$$f(u, v)\, g(r, s) = h(x, y),$$

where x and y are quadratic polynomials in u, v, r, s. Gauss proved that this can always be done. The formulas above can mislead one in to guessing that this is simply a question of finding the right generalization, but that is far from the truth. (1), (2), (3) and (4) are explicit only because they are very special cases in the theory.

In Gauss' proof he had to prove that various other equations could be solved in integers in order to find h and the quadratic polynomials x and y. This was so complicated that some of the intermediate formulas took two pages to write down, and are very difficult to make sense of. See article 234 and beyond in Gauss' book *Disquisitiones Arithmeticae* (1804).

5 DIRICHLET COMPOSITION

Dirichlet claimed that when he was a student working with Gauss, he slept with a copy of *Disquisitiones* under his pillow every night for three years. It worked, and Dirichlet found a way to better understand Gauss' proof of composition, which amounted to a straightforward algorithm to determine the composition of two given binary quadratic forms f and g of the same discriminant. The key was to prove that there exist quadratic forms, $F(x, y) = ax^2 + bxy + cy^2$, equivalent to f, and $G(x, y) = Ax^2 + bxy + Cy^2$, equivalent to g, for which $(a, A) = 1$. Notice that the middle coefficients of F and G are the same. Since these have the same discriminant, we deduce that ac = AC and so there exists an integer h for which

$$F(x, y) = ax^2 + bxy + Ahy^2 \text{ and } G(x, y) = Ax^2 + bxy + ahy^2.$$

Then

$$H(ur - hvs,\ aus + Avr + bvs) = F(u, v)G(r, s)$$

$$\text{where } H(x, y) = \mathrm{a}Ax^2 + bxy + hy^2.$$

Dirichlet went on to interpret this in terms of what today we would call ideals, and this led to the birth of modern algebra as established by Dedekind. In this theory, one is typically not so much interested in the identity, writing H as a product of f and g (which is typically very complicated and none too enlightening), but rather in determining H from f and g. There is an important interpretation in terms of group theory, but it would take us too far afield for this article.

6. BHARGAVA COMPOSITION[1]

Let us begin with one further explicit composition, a tiny variant on (4) (letting $s \to -s$ there):

$$(au^2 + 2Buv + cv^2)(ar^2 - 2Brs + cs^2) = x^2 + (ac - B^2)y^2$$

where $x = aur + B(vr - us) - cvs$ and $y = us + vr$.

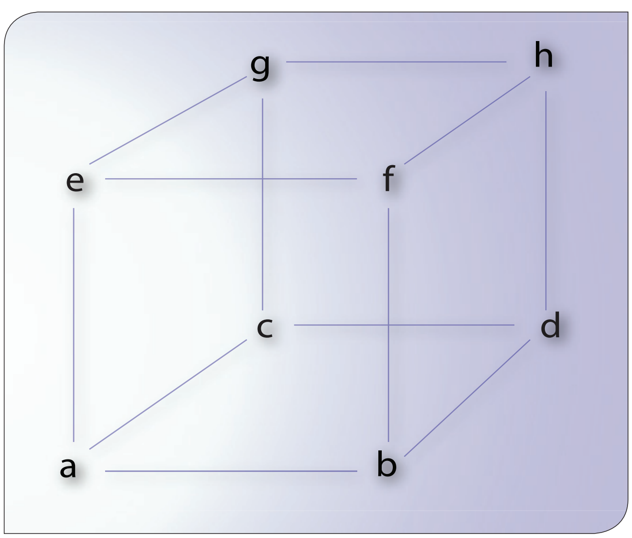
Combining this with the results of the previous section suggests that if the discriminant d is divisible by 4 (which is equivalent to b being even), then

$$F(u, v)\, G(r, s)\, H(m, n) = P(x, y) \qquad (6)$$

where $P(x, y) = x^2 - \frac{d}{4}y^2$ and x and y are cubic polynomials in m, n, r, s, u, v. Analogous remarks can be made if the discriminant is odd.

In 2004 Bhargava came up with an entirely new way to find all of the triples F, G, H of binary quadratic forms of the same discriminant for which (6) holds: We begin with a 2-by-2-by-2 cube, the corners of which are labeled with the integers a, b, c, d, e, f, g, h.

There are six faces of a cube, and these can be split into three parallel pairs. For each such pair, consider the pair of 2-by-2 matrices given by taking the entries in each face, those entries corresponding to opposite corners of the cube, always starting with a.



Hence we get the pairs

$$M_1(x,y) := \begin{pmatrix} a & b \\ c & d \end{pmatrix} x + \begin{pmatrix} e & f \\ g & h \end{pmatrix} y = \begin{pmatrix} ax+ey & bx+fy \\ cx+gy & dx+hy \end{pmatrix},$$

$$M_2(x,y) := \begin{pmatrix} a & c \\ e & g \end{pmatrix} x + \begin{pmatrix} b & d \\ f & h \end{pmatrix} y = \begin{pmatrix} ax+by & cx+dy \\ ex+fy & gx+hy \end{pmatrix},$$

$$M_3(x,y) := \begin{pmatrix} a & b \\ e & f \end{pmatrix} x + \begin{pmatrix} c & d \\ g & h \end{pmatrix} y = \begin{pmatrix} ax+cy & bx+dy \\ ex+gy & fx+hy \end{pmatrix},$$

where we have, in each, appended the variables, x, y, to create matrix function of x and y. The determinant, $-Q_j(x, y)$, of each $M_j(x, y)$, is a quadratic form in x and y (the *determinant* of a 2-by-2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by $ad - bc$). Incredibly Q_1, Q_2 and Q_3 all have the same discriminant and their composition equals P, just as in (6).

Let's work though an example: Plot the cube in 3-dimensions and label each corner with its Cartesian co-ordinates (x,y,z), each 0 or 1, and then label this corner with the binary number, $4x + 2y + z$, squared. Hence,

$$a, b, c, d, e, f, g, h = 2^2, 6^2, 0^2, 4^2, 3^2, 7^2, 1^2, 5^2,$$

leading to three binary quadratic forms of discriminant $-7 \cdot 4^4$:

$$Q_1 = -4^2(4x^2 + 13xy + 11y^2),\ Q_2 = -2^2(x^2 - 2xy + 29y^2),$$
$$\text{and } Q_3 = 4^2(8x^2 + 5xy + y^2).$$

After some work one can verify that

$$Q_1(m, n)\, Q_2(r, s)\, Q_3(u, v) = 4(x^2 + 4^3 \cdot 7y^2),$$

where x and y are the following cubic polynomials in m, n, r, s, u, v:

$$x = 8(-11mru - 3mrv + 25msu + 17msv - 17nru - 4nrv + 59nsu + 32nsv)$$

and $y = mru + mrv + 21msu + 5msv + 3nru + 2nrv + 31nsu + 6nsv$.

1. There is no Nobel Prize in mathematics; the nearest equivalent is the Fields Medal, though this is only given to people 40 years of age or younger. They are awarded every four years, up to four each time, the most recent being on August 13th, 2014, in Korea. One of the laureates was Manjul Bhargava, recognized for a body of work which begins with his version of composition, as discussed here, and allows him to much better understand many classes of equations, especially cubic. Bhargava was born in Hamilton, Ontario, and was the first Canadian to receive this most prestigious award.

Bhargava proves his theorem, inspired by a 2-by-2-by-2 Rubik's cube. The idea is to apply an invertible linear transformation simultaneously to a pair of opposite sides. For example, if one applies an invertible linear transformation to the first pair of sides, then the binary quadratic form Q_1 is transformed in the usual way, whereas Q_2 and Q_3 remain the same. One can do this with any pair of sides. This allows one to proceed in "reducing" the three binary quadratic forms to equivalent forms that are easy to work with (rather like in Dirichlet's proof). This brings to mind the twists of the Rubik's cube, though in that case one has only finitely many possible transformations, whereas here there are infinitely many possibilities!

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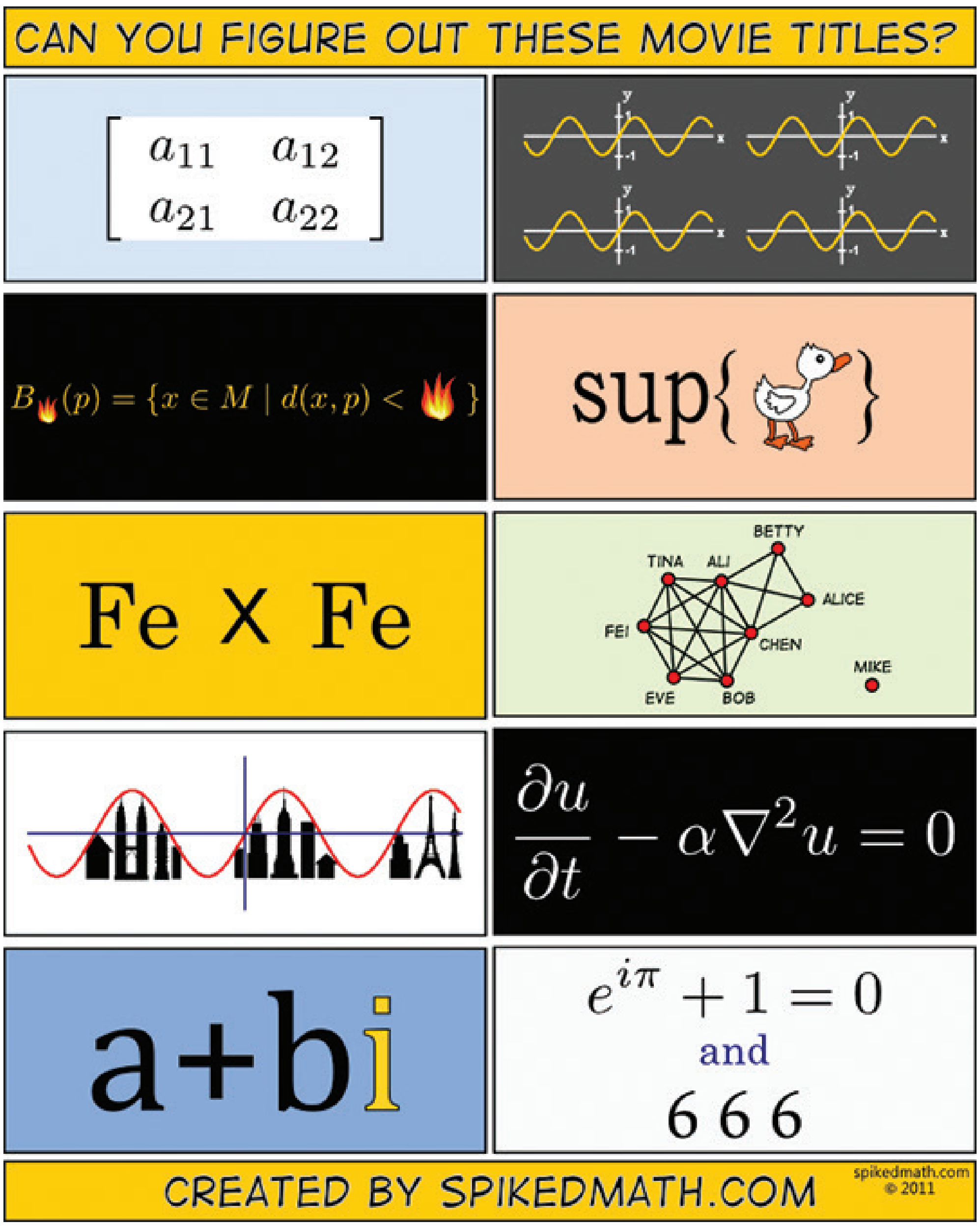
[1] Manjul Bhargava, *Higher Composition Laws I: A New View on Gauss Composition, and Quadratic Generalizations*, Annals of Mathematics **159** (2004), 217-250.

[2] Karl Frederich Gauss, *Disquisitiones Arithmeticae* (1804).

Departement de mathematiques et de statistiques, Universite de Montreal, Montreal QC H3C 3J7, Canada.

andrew@dms.umontreal.ca



3D PRINTING IN MATHEMATICS

MICHAEL ANTON AND MICHAEL P. LAMOUREUX

M. Anton is an independent hardware/firmware designer in Edmonton. M. Lamoureux is a Professor at the University of Calgary.

Introduction

3D printing is all the rage these days: we see it on TV shows like *The Big Bang Theory*, when Howard prints a little doll-like image of himself; on the news where we learn medical researchers can print an artificial bone or liver for use in transplants; or on the Internet where enthusiasts show how to print out a working gun.

So how hard is 3D printing? And can we make use of it in mathematics? It turns out it is pretty easy once you have the printer, and yes, it can be used for mathematics. For example, Figure 1 shows the 3D printout of the well-known trefoil knot, which we are used to seeing as only a 3D image on a computer screen (such as in Figure 2). The two authors worked this out in a couple of hours one afternoon, at the 2013 Mini Maker Faire in Calgary.

Fig. 1: A real, printed 3D trefoil knot, in plastic.

How it works

First, you need a 3D printer. It's a lot like an inkjet printer, except it uses melted plastic instead of ink and the printing head can move in three dimensions to create a solid object. For a couple of thousand dollars, you can buy one for yourself or your school. You can also rent time on someone else's machine, or contract them to print for you. There is a choice of material: typically one uses either PLA (polylactic acid) bioplastic derived from environmentally-friendly food starch, or ABS (Acrylonitrile butadiene styrene) plastic, made of more robust styrene.

The first author decided to build his own 3D printer, based on the MendelMax V1.5 plans publicly available on the Internet, which he modified to make a bigger, more reliable device.

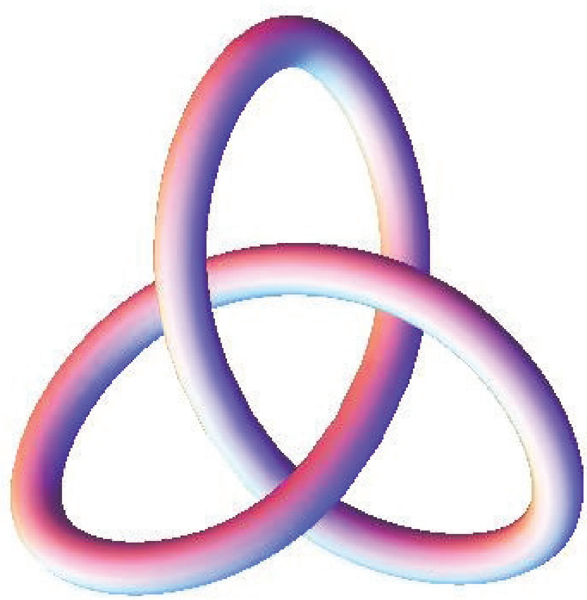


Fig. 2: A computer image of a trefoil knot.

After a few hundred hours of design and construction, Figure 3 shows the result. On the printer (which is about the size of a microwave oven) you can see the blue platform where the grey 3D object is being printed. Above the platform is the printer head extruder and hot end, which moves back and forth along the chrome-colored rods. The black beams and many 3D-printed components form the main structure that houses all the moving parts. Hidden underneath are motors, fans and a CPU that controls the whole device.

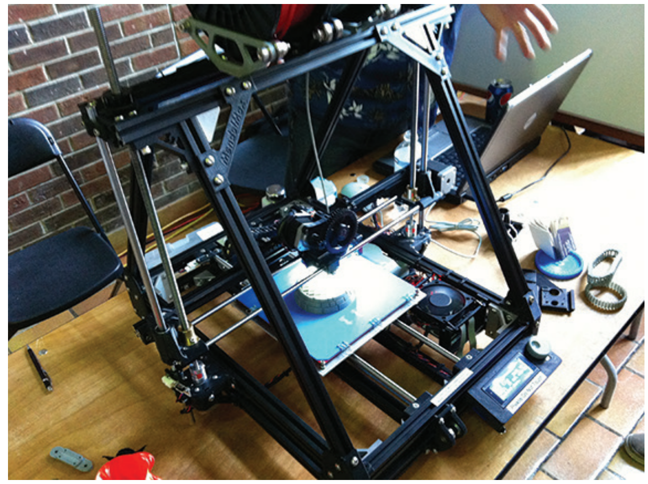


Fig. 3: A 3D printer, built by the first author.

Once you have access to a printer, how do you print your mathematics? It is not as simple as plugging in your laptop and clicking "print," but it is not much harder than that. 3D printers understand something called G-code, so the main process is to get your 3D mathematical object into G-code and pass that to the printer. There is an intermediate stage, where the mathematical object to be created is represented in the Standard Tessellation Language (STL format), which expresses the 3D object as a collection of triangles covering the object to a certain degree of resolution.

The basic steps in printing a mathematical object are:

- create a 3D image in a mathematical software package;
- export to an STL file;
- view the STL, fix errors, and create G-code;
- transfer the G-code to the printer;
- print!

For instance, here's how we printed the trefoil knot. Using Mathematica®, the command "KnotData" is used to create a 3D knot. We exported the trefoil knot to an STL file using the single command **Export["trefoil.stl",KnotData["Trefoil"]]**.

We then used the program *Repetier-Host* to view the STL file in a nice graphical user interface. Repetier- Host showed the trefoil knot we created was very small, so we scaled it up by 100x, to make something printable. We then sent it to the *netfabb* cloud service which automatically corrects any errors in the STL file. The result from netfabb was run through *slic3r* (included with Repetier-Host) to create the G-code. We chose the option to slice "with support" which created a physical scaffolding for the trefoil knot as it printed. We transferred the G-code into an SD memory card, plugged the card into the printer, and voila, it printed!

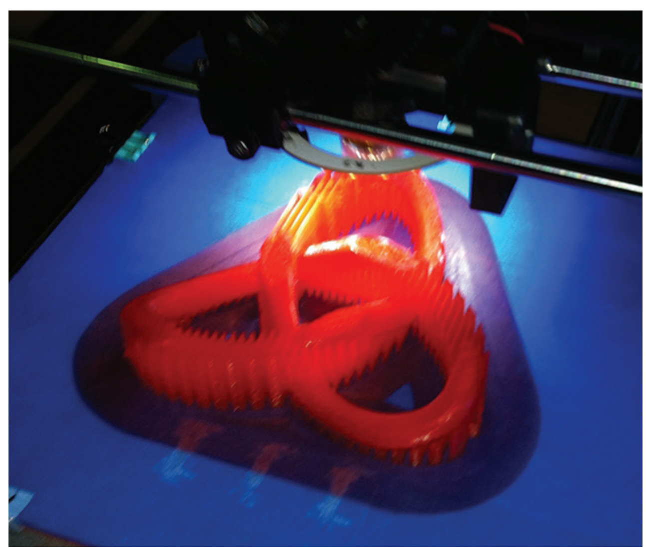


Fig. 4: The knot in the process of printing.

Figure 4 shows the knot as it is being printed, and Figure 5 shows the final result – with the scaffolding clearly visible underneath the knot. This scaffolding easily peels off to leave just the knot itself, as shown in Figure 1.

Incidentally, we did try printing the knot without scaffolding, but the pieces eventually fell apart during the printing process, as shown in Figure 6.

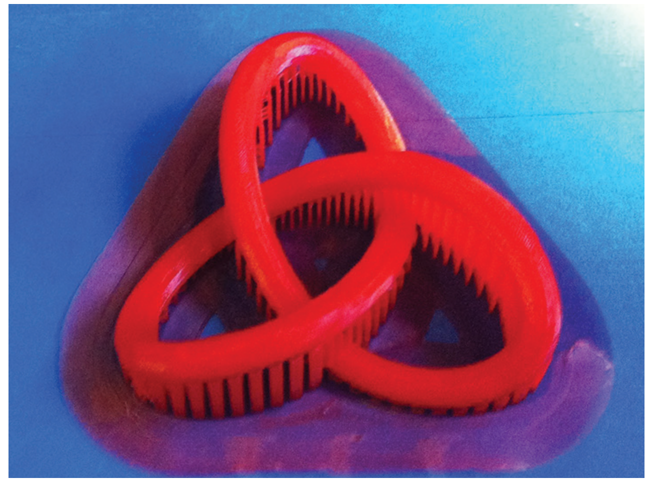


Fig. 5: The completed knot, on the printing platform.

Going further

We started this short project with the idea that we want to print some fractals, such as the Mandelbrot set or the Hofstadter butterfly. This is tricky, as fractals are very complex shapes and it is not clear how the STL file format can capture this complexity, and translate it into something that the printer can handle. They also tend to be very thin, so how do we make something solid and printable? Some people have already figured this out. For instance, Figure 7 shows a fractal vase printed by the folks at Machina Corp, where the cross section is the familiar Koch curve. It's a fascinating little vase, and people see it and like it even without knowing the math behind it. Perhaps this can be a challenge to you – to turn a mathematical image into a nice friendly object like a vase!

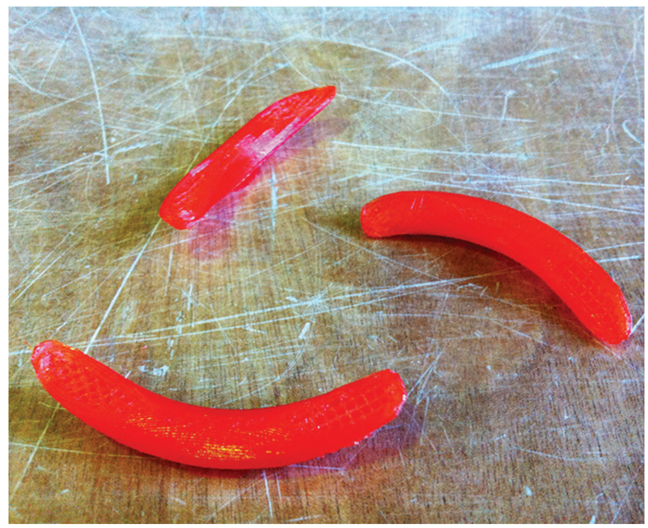


Fig. 6: An unsuccessful 3D print.

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Machina Corp: *http://machinacorp.ca*
Mathematica: *http://www.wolfram.com/Mathematica*
Netfabb: *http://cloud.netfabb.com*
Repetier-Host: *http://www.repetier.com*

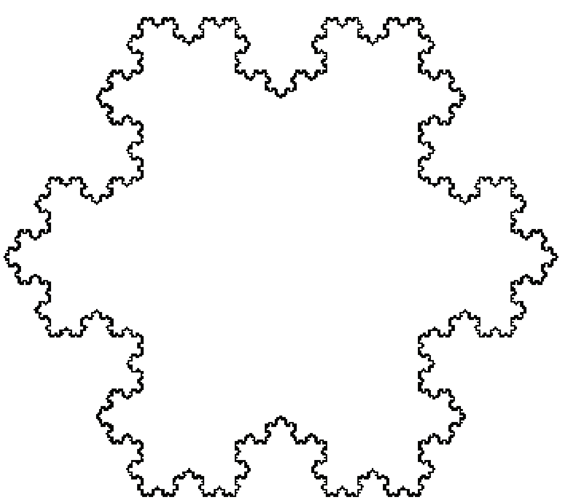

Fig. 7: A fractal vase, from the 3D printer.

Knot Just Another Math Article

An Introduction to the Mathematics of Knots and Links

BY ROBIN KOYTCHEFF

Take a string, such as a loose shoelace, tie it up in some manner and then tape the two ends together to form a closed loop. A thin cable that can be clipped to itself will also work. Perhaps you will tie either the left or middle loop shown in Figure 1. Without untaping the ends or cutting the string in any way can you untangle either of these loops to get an unknotted circle, as in the rightmost loop in Figure 1? You might be tempted to say no, but would you be willing to bet a million dollars that it is impossible to untangle these knots?

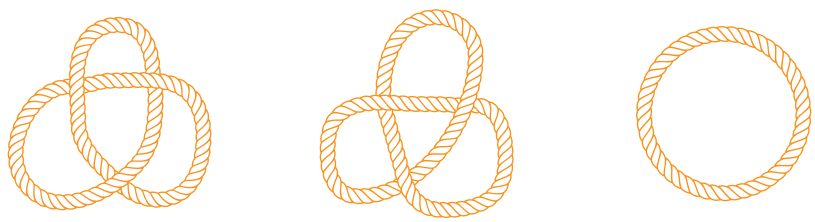

Figure 1. A few simple knots. The one on the right is called the *unknot*. Can you untangle either of the other two to get the unknot?

Next, consider the knots in Figure 2. Again, by moving the strings around in space, but without cutting, can you make some of these knots into the same knot? Is any of them the unknot? Depending on the availability of nearby strings and tape, you might prefer to attempt this puzzle with a pen and paper, redrawing the pictures in Figure 2 and drawing what they would like after you moved certain strands around in space. We'll guide you through some examples below, but you might enjoy trying it on your own without any hints. If you want to start with a small hint, there is at least one pair of knots which are "the same knot."

As an example of how to untangle one knot into another, start with the second knot shown in Figure 2. Take the red strand between the two dots shown in the first picture in Figure 3 and drag it over and to the left to where the dotted arc is. The middle picture shows the resulting knot, which you can reshape into the third knot in Figure 2 by wiggling the string a bit. Thus the second and third knots in Figure 2 are "the same knot."

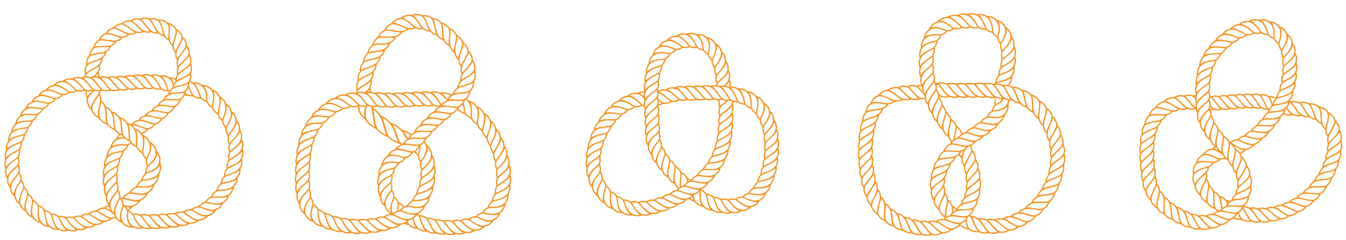

Figure 2. Some more knots. Which of these are the same?

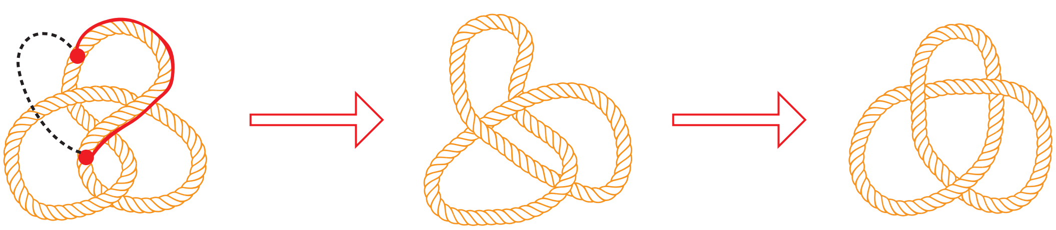

Figure 3. "Untangling" a knot. Notice the rightmost picture has fewer crossings than the leftmost picture.

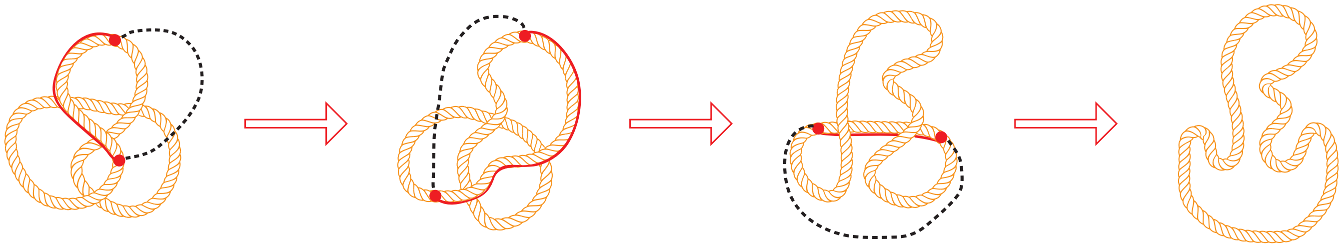

Figure 4. "Untangling" a knot into the unknot.

In Figure 4, we show that the rightmost picture in Figure 2 is really just a complicated picture of the unknot. So far our untangling has shown that there are at most four different knots in Figure 2. It turns out that the first and fourth knots are "the same," though seeing this may be bit more difficult than the above examples. So there are actually, at most, three different knots in Figure 2. Are all of those different? It might seem impossible to untangle the knot in Figure 3 to the unknot, but before you jump to this conclusion consider that both of the knots in Figure 5 can be untangled to get the unknot.

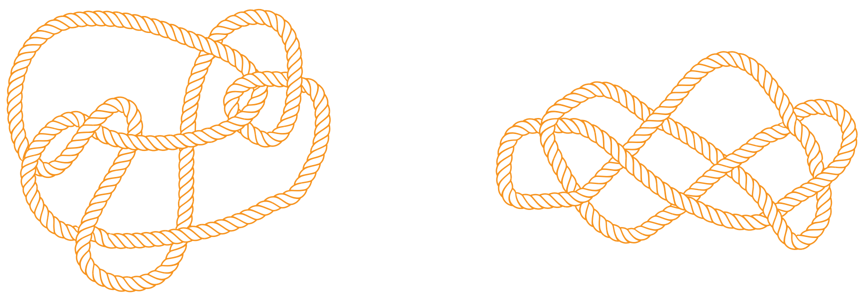

Figure 5. Two complicated pictures of the unknot.

This leads us to the main underlying problem of knot theory, nowadays a mathematical subject in its own right. The problem is to classify all knots, or even just tell when two arbitrary knots are different knot types. In the 19th century, Lord Kelvin conjectured that different knot types corresponded to different atoms in the periodic table. This theory turned out to be false, but led P. G. Tait to begin the systematic tabulation of knots. There are infinitely many knots, just like there are infinitely many whole numbers. The use of computers has allowed mathematicians to tabulate the first six billion knots, but we do not yet have a systematic way of understanding all knots. The situation is analogous to studying numbers. Even though we know how to write down all the numbers, there are still patterns among them (such as properties of prime factorizations) that we do not fully understand. Arguably, knots are even more difficult because it is easier to write down the first six billion numbers than the first six billion knots. Moreover, knot theory has applications in studying DNA molecules [Ada94, Mur96] (and perhaps others still to be discovered). Here, we will focus on purely mathematical aspects of the subject.

We now turn to the problem of distinguishing knots. Start with a *knot diagram or planar projection* of a knot, meaning a picture of the knot lying on a table, with over/under- crossings, such as any of the pictures above. Suppose you had a function on all such diagrams. By a function we mean a "machine" that takes in such a diagram as input and gives a number as output. If such a function gives the same number for any two pictures of the same knot type, we call it a *knot invariant*.

Let's try to find a knot invariant. We want to associate a number to each picture, so we'll try counting something. As a first guess, we might try counting the crossings in a diagram. However, this is not an invariant: by untwisting the knot on the left in Figure 6, we see that it is same knot as the unknot, whose standard diagram is shown on the right; however, the diagram on the left has one crossing, whereas the diagram on the right has no crossings. A slicker variation would be to count the "minimum number of crossings in any planar projection of this knot type." By definition, this does give the same number for any two diagrams of the same knot type!

But this is not easy to compute. To know that a diagram has a minimal number of crossings, we would have to know that it is impossible to lower this number by further untangling. In some cases, you may feel convinced, but would you bet a million dollars? Admittedly, we would for the diagram on the right in Figure 6, which has no crossings. So the minimal crossing number for the unknot is zero, but that's not useful so far.

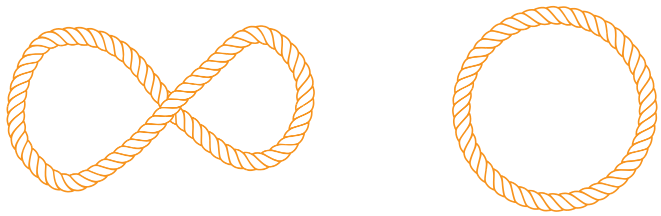

Figure 6. Two planar projections of the unknot, with different numbers of crossings.

If we desperately wanted a knot invariant, we could associate any fixed number, such as 17, to every diagram. Any constant function like this qualifies as a knot invariant and is easy to compute, but it is useless for distinguishing knots.

Rather than pursuing knot invariants directly, we will consider a similar setup where the first interesting invariant is easier to describe. Consider two strings tied up into loops in space. Such objects are called *links (of two components)*. The simplest two-component link, the *unlink* is shown below on the left. Each component is drawn in a different color; this helps clarify that there are two separate strings, which can be difficult to see in more complicated examples such as in Figure 12. It sure looks like you can't pull apart the two strings on the right of Figure 7 without cutting the strings. But would you bet a million dollars that it's impossible?

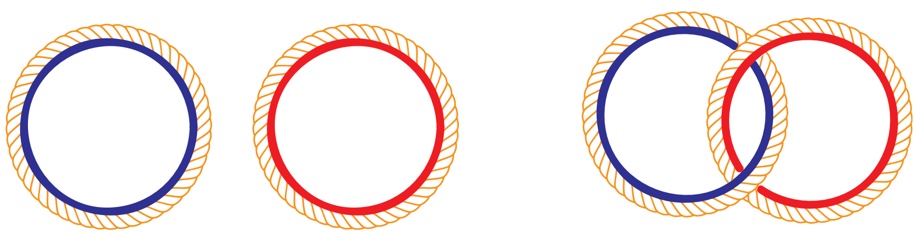

Figure 7. Two examples of links of two components.

Soon we'll see that you should bet that million dollars, should the opportunity arise. We'll describe an invariant of two-component links that takes different values on the two links above. First, notice that for each of the two components in the link, there are two ways of going around the loop. Pick one such orientation and mark it with an arrow as in Figure 8.

We will now define the linking number ℓ as follows. We will count the number of times the two components cross, but some crossings will count as +1, and others will count as −1. Specifically, as in Figure 9, a crossing as on the left will be called *positive* and count as +1,

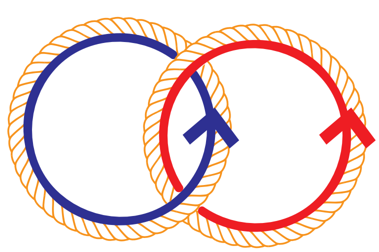

Figure 8. A link with an orientation on each of its components.

while a crossing as on the right will be called *negative* and count as -1. Note that we will

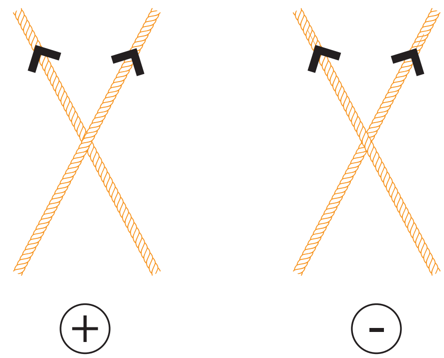

Figure 9. A positive and a negative crossing.

not count a crossing where both strands are on the same component. (This is where drawing the two components in different colors can be useful.) The colors of the strands do not change the sign of the crossing; the only point is that we do not count crossings of strands of the same color. See Figures 10 and 11.

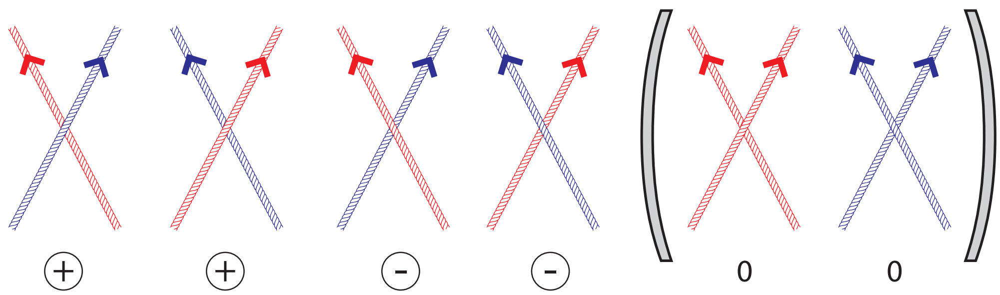

Figure 10. How to count crossings for a two-component link, with the different link components shown in two different colors.

Now, add up all the $+1$'s and -1's. It turns out that the result will always be an even integer, so divide this number by two; this is the linking number ℓ. This invariant was studied by Gauss in the early 19th century via calculus and thus, ℓ is sometimes called the Gauss linking number (and the integral that Gauss wrote down to compute ℓ is called the Gauss linking integral).

To verify that this number is independent of the planar projection, i.e., an *invariant* of links, we would use a theorem of Reidmeister from the 1930s. This theorem says that if

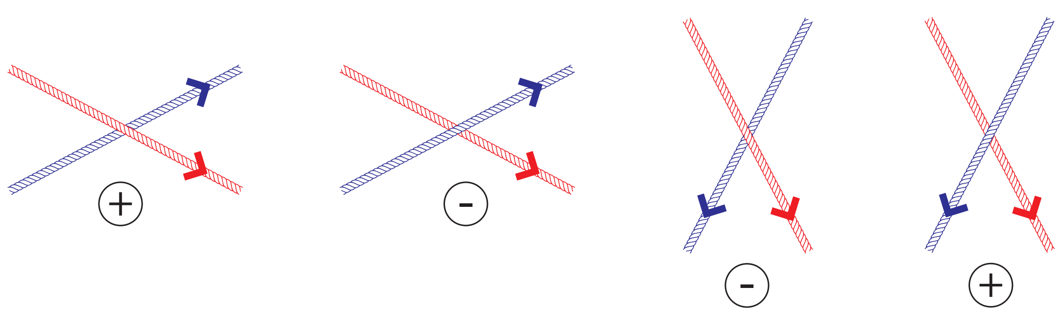

Figure 11. Some more examples showing how to count crossings for a two-component link. The sign for each picture here can be deduced by rotating the picture so that the arrows point upward as in Figure 10.

this count ℓ is unchanged under three simple types of moves of the strands, then ℓ is an invariant. The details are slightly beyond the scope of this article but may be found in books such as [Ada94, Mur96].

Changing the orientation on either component will multiply the resulting ℓ by -1. (As an exercise, you can try to convince yourself of this.) So actually, ℓ is an invariant of *oriented* two-component links, i.e., links with a chosen orientation on each component. Still, this helps us distinguish some links. For example, the ℓ of the projection of the unlink shown at the left in Figure 7 is zero, regardless of the choice of orientations: there are no crossings to count. If we believe that ℓ is independent of the planar projection, then we conclude that it is zero for any projection of the unlink. Then if ℓ is nonzero on some other link, it will still be nonzero after multiplying by -1, i.e., it will be nonzero regardless of our choice of orientations. Thus, that link cannot be the unlink. In particular, we can now deduce that the link on the right in Figure 7 (called the *Hopf link*) cannot be the unlink: depending on how you oriented the two circles, you'll get $+1$ or -1 for the linking number. (With the choices shown in Figure 8, we get -1.)

If you are unimpressed by having distinguished the links in Figure 7, you may want to try computing the linking number of some more complicated links, where it may not be so obvious that you can't untangle to get the unlink. After calculating the linking numbers,

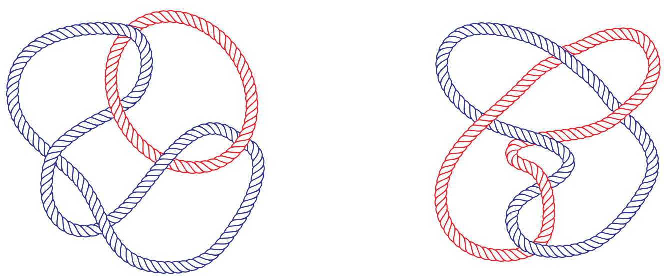

Figure 12. More examples of two component links.

you may also find that you can distinguish these links from each other, as well as from the Hopf link. Finally, we point out that even this invariant has its limitations. As you can check, the linking number of the link in Figure 13 is zero. However, this link cannot be untangled. Explaining this is beyond the scope of this article, so we leave the interested reader to explore this, as well as constructions of interesting invariants of knots, in future studies.

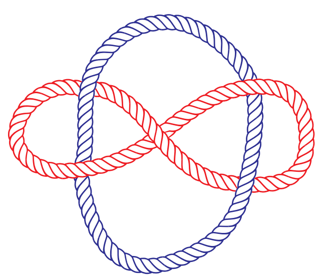

Figure 13. The Whitehead link, which has linking number zero, but is not the unlink.

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Mathematics and Statistics, University of Victoria, Victoria, BC, Canada
E-mail address: rmjk@uvic.ca

A RIGHT WAY TO USE INDUCTIVE REASONING

BY LINYI CHENG, COLONEL BY SECONDARY SCHOOL, OTTAWA

Linyi Cheng wrote this article while he was a high school student at Colonel By Secondary School, Ottawa, and is currently studying at the University of Waterloo.

ABSTRACT

This article aims to find and prove, with induction, the formula that determines the maximum total number of regions formed by an inscribed convex polygon and its intersecting diagonals.

INTRODUCTION

Inductive logic – the reasoning that *Event X* will occur because it occurred before – has long been used throughout human civilisation. Before the dawn of modern science, our ancestors debated whether or not we could ever be certain that the sun would rise again the next day. Some philosophers and logicians reasoned that because it had risen every day in the past, it would, and surely must, rise again tomorrow. At first, we might think that this reasoning is a bit silly… but wait, we reason like this in mathematics as well!

Take the following sums below as an example. What is their pattern?

$1^2 = 1$
$2^2 = 1+3$
$3^2 = 1+3+5$
$4^2 = 1+3+5+7$

We quickly deduce from the given sums that $n^2 = 1+ 3 + 5 +...+ (2n - 1)$. Although we are correct, we, like those who believed that the sun would rise again simply because of past observations, have drawn this conclusion only from the results of the first four sums. When applying inductive logic to conjecture a formula, one must ensure that the formula works not only for the terms used to conjecture it, but also for all the subsequent terms. We will soon encounter such an example.

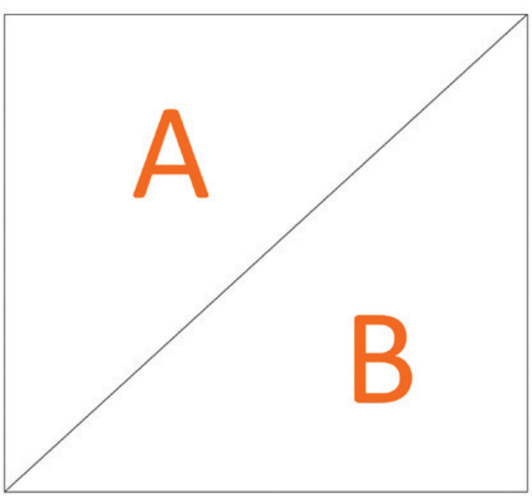


Figure 1 There are two regions: (A) and (B).

Let's proceed to our objective: can we find a formula that determines the maximum total number of regions formed from the intersecting diagonals in a convex polygon plus the number of regions formed by its circumscribing circle (which we shall represent as $C(n)$)? A region refers to a section enclosed by a number of vertices such that it does not contain any smaller sections. For example, in Figure 1, there are two regions, (A) and (B), whereas (A+B) is not a region because it is formed from combining (A) and (B) together.

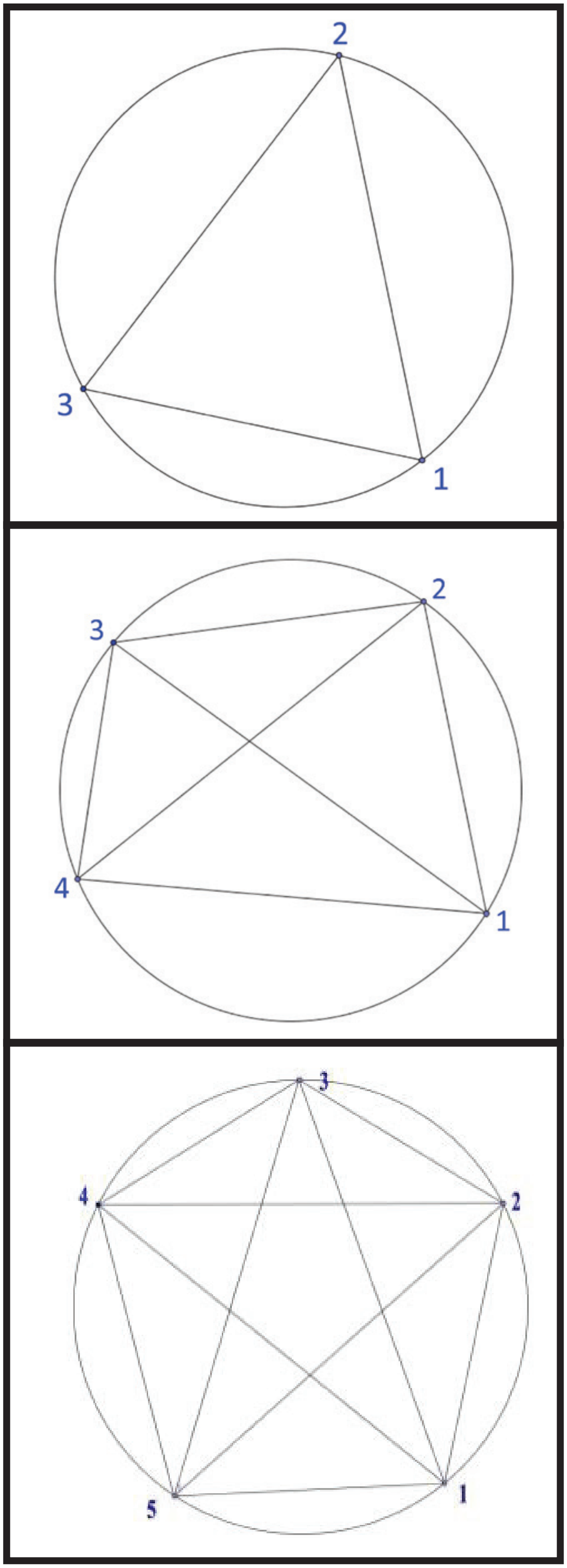


Figure 2 Including the regions formed from their circumscribing circles, a triangle forms four regions, a quadrilateral eight regions, and a pentagon 16 regions.

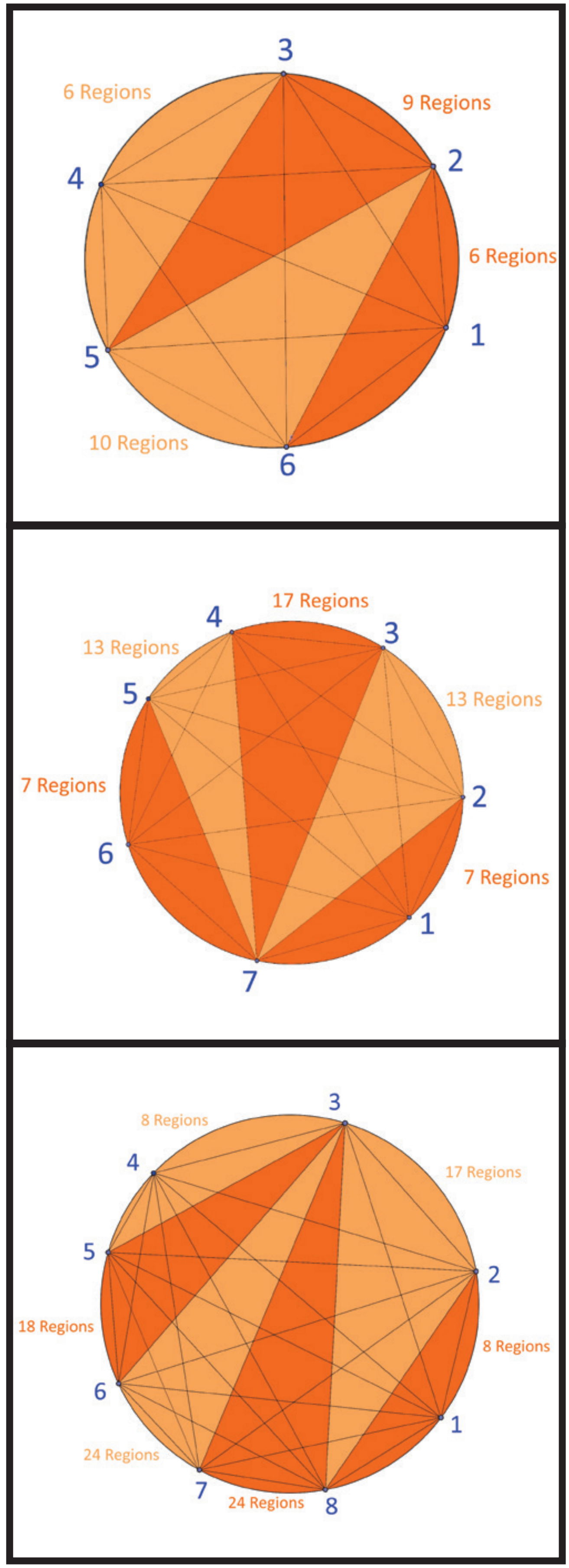


Figure 3 Including the regions formed from their circumscribing circles, a hexagon forms 31 regions, a heptagon 57 regions, and an octagon 99 regions. None of these are powers of two.

In Figure 2, we have three polygons whose values of $C(n)$ follow successive powers of two, so it may be reasonable to believe that the formula is $C(n)=2^{n-1}$, where n is the number of vertices the polygon has. Being diligent problem solvers, we check if this formula is indeed valid, by examining several other values of n and their corresponding $C(n)$ values. The results are shown in Figure 3.

The formula breaks down! Although it is true for the first three inscribed polygons, it is not true for the ones afterwards. This is an example in which a formula drawn from the first couple of terms in a sequence does not correctly describe the later terms. In this article, we shall find the real formula that describes the relationship between n and $C(n)$ and to validate its legitimacy, we shall prove that the formula is indeed correct.

CONJECTURING A FORMULA

The values of n with their corresponding $C(n)$ values are shown in Table 1 and they were experimentally determined by drawing inscribed polygons with non-concurrent diagonals. We assume that there is a polynomial relationship between n and $C(n)$ and we determine the degree of the polynomial by finding the $C(n)$ values' progressive differences until they are equal, which is shown in Table 2.

n : NUMBER OF VERTICES:	3	4	5	6	7	8
$C(n)$: MAX. NUMBER OF REGIONS:	4	8	16	31	57	99

Table 1 Maximum number of regions formed by an inscribed convex polygon and its intersecting diagonals.

$C(n)$ VALUES	FIRST DIFFERENCE	SECOND DIFFERENCE	THIRD DIFFERENCE	FOURTH DIFFERENCE
4	4	4	3	1
8	8	7	4	1
16	15	11	5	
31	26	16		
57	42			
99				

Table 2 The successive differences in $C(n)$ values.

Since the fourth differences are equal, this is a quartic polynomial of the form $C(n) = an^4 + bn^3 + cn^2 + dn + e$. We determine its coefficients by first substituting values of n with their corresponding $C(n)$ values, which are shown in Eqns. 2.1 - 2.5. Then, by elimination, we determine that $a = \frac{1}{24}$, $b = -\frac{1}{4}$, $c = \frac{23}{24}$, $d = -\frac{3}{4}$, and $e = 1$.

Therefore, $$C(n) = \frac{1}{24}n^4 - \frac{1}{4}n^3 + \frac{23}{24}n^2 - \frac{3}{4}n + 1 \quad (1)$$

$$3^4a + 3^3b + 3^2c + 3d + e = 4 \quad (2.1)$$

$$4^4a + 4^3b + 4^2c + 4d + e = 8 \quad (2.2)$$

$$5^4a + 5^3b + 5^2c + 5d + e = 16 \quad (2.3)$$

$$6^4a + 6^3b + 6^2c + 6d + e = 31 \quad (2.4)$$

$$7^4a + 7^3b + 7^2c + 7d + e = 57 \quad (2.5)$$

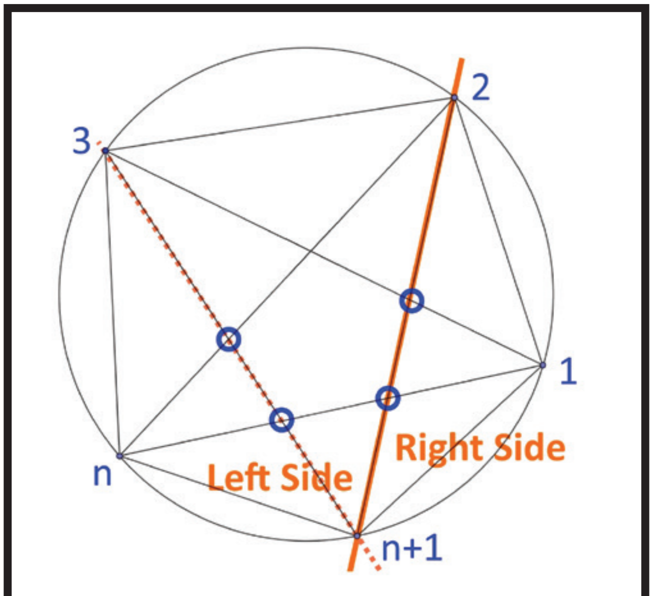
It is interesting to note that if this formula were plotted on a Cartesian Plane, every integer x outputs an integer y – this formula generates lattice points!

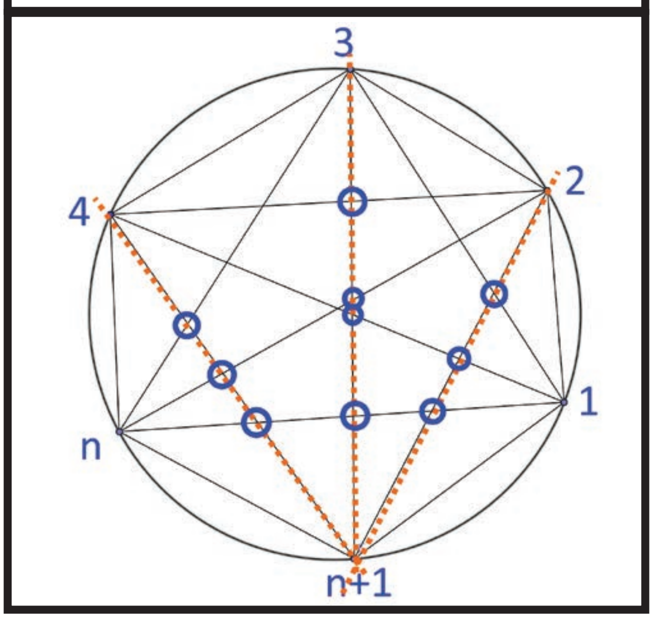
BEGINNING THE PROOF

We will prove our conjectured formula using induction, but to do so we need to understand how new regions are formed when a vertex is added to an existing inscribed polygon. Let's investigate this by adding vertices to an existing inscribed quadrilateral and pentagon.

ADDING A VERTEX TO AN EXISTING INSCRIBED QUADRILATERAL (n = 4)

Let a vertex be denoted by $V_{\text{vertex number}}$ and a diagonal connecting V_n and V_m be denoted by $D_{n,m}$. Also, let all the vertices be *orderly numbered* such that V_{n+1}'s adjacent vertices are V_1 and V_n. Without loss of generality, we assume that $D_{n+1,2}$ divides the existing quadrilateral into a left side and a right side, as shown in Figure 4. Any diagonal connecting vertices from $D_{n+1,2}$'s left side to vertices on its right side intersects $D_{n+1,2}$. Since there are two vertices on its left side, V_3 and V_n, and one on its right side, V_1, and because diagonals connect every vertex from its left side to every vertex on its right side, $D_{n+1,2}$ intersects 2 x 1 = 2 diagonals, forming three new regions (since the number of new regions formed is one more than the number of diagonal intersections). Similarly, there is one vertex on $D_{n+1,3}$'s left side (V_n), and two vertices on its right side (V_1 and V_2). So, $D_{n+1,3}$ also intersects two diagonals and forms three new regions. It is interesting that because $D_{n+1,1}$ and $D_{n+1,n}$ both have three vertices on their left side, zero on their right side and vice versa respectively, they both intersect 3 x 0 = 0 diagonals and form one new region. In total, eight new regions are formed when a vertex is added to an existing inscribed quadrilateral.





TOP. Figure 4 Adding a vertex when n=4. The blue circles highlight points of intersections between the diagonals connected to V_{n+1} and other existing diagonals.

BOTTOM. Figure 5 Adding a vertex when n=5.

ADDING A VERTEX TO AN EXISTING INSCRIBED PENTAGON (n= 5)

$D_{n+1,2}$ and $D_{n+1,4}$ both have three vertices on their left side, one on their right side and vice versa respectively, which means that they both intersect 3 x 1 = 3 diagonals and form four new regions.

Also, a diagonal that connects two vertices both located on the same side of another diagonal does not intersect the latter diagonal and so does not form new regions.

$D_{n+1,3}$ has two vertices on its left and right side so it intersects 2 x 2 = 4 diagonals and forms five new regions. $D_{n+1,1}$ and $D_{n+1,n}$ both intersect 4 x 0 = 0 diagonals and form one new region. In total, 15 new regions are formed.

From our investigation, we found that the total number of new regions formed by adding V_{n+1} to the existing inscribed polygon is the sum of the number of new regions formed by $D_{n+1,j}$, for j in the range of 1 to n. Since $D_{n+1,j}$, for the same range of j as previously mentioned, has $n - j$ vertices on its left side and $j - 1$ vertices on its right side $D_{n+1,j}$ intersects $(n - j)(j - 1)$ diagonals and forms $(n - j)(j - 1) + 1$ new regions.

Let $\Delta(n+1)$ denote the total number of new regions formed when a vertex is added to an existing inscribed polygon. Therefore:

$$\begin{aligned}
\Delta(n-1) &= \sum_{j=1}^{n}[(n-j)(j-1)+1] \\
&= \sum_{j=1}^{n} jn - \sum_{j=1}^{n} n - \sum_{j=1}^{n} j^2 + \sum_{j=1}^{n} j + \sum_{j=1}^{n} 1 \\
&= [n + 2n + \ldots + (n-1)n + (n)n] - n^2 - [1^2 + 2^2 + \ldots + (n-1)^2 + (n)^2] \\
&\quad + [1 + 2 + \ldots + (n-1) + (n)] + n \\
&= \frac{n^2(n+1)}{2} - n^2 - \frac{n(n+1)(2n+1)^*}{6} + \frac{n(n+1)}{2} + n \qquad {}^*\text{from formula for square pyramidal numbers} \\
&= \frac{n^3+n}{2} - \frac{n(n+1)(2n+1)}{6} + n \\
&= \frac{3n^3+3n}{6} - \frac{2n^3+3n^2+n}{6} + \frac{6n}{6} \\
&= \frac{n^3-3n^2+8n}{6} \qquad (3)
\end{aligned}$$

Now that we have our formula for $\Delta(n+1)$, we are ready to prove with induction that our original quartic formula is correct for all values of n.

FINISHING THE PROOF

We will assume that $C(k)$ is given by $\frac{1}{24}k^4 - \frac{1}{4}k^3 + \frac{23}{24}k^2 - \frac{3}{4}k + 1$, where $k \in Z^+ \mid k \geq 3$ and represents the number of vertices the inscribed polygon has.

We calculated above in Eqn. 3 that $\Delta(k+1) = \left(\frac{k^3 - 3k^2 + 8k}{6}\right)$, which means that we can show that:

$$\begin{aligned}
C(k+1) &= C(k) + \Delta(k+1) \\
&= \frac{1}{24}k^4 - \frac{1}{4}k^3 + \frac{23}{24}k^2 - \frac{3}{4}k + 1 + \frac{k^3 - 3k^2 + 8k}{6} \\
&= \frac{1}{24}k^4 - \frac{6}{24}k^3 + \frac{23}{24}k^2 - \frac{18}{24}k + \frac{24}{24} + \frac{4k^3 - 12k^2 + 32k}{24} \\
&= \frac{k^4 - 2k^3 + 11k^2 + 14k + 24}{24}
\end{aligned}$$

Now, let's verify that this value of $C(k+1)$ agrees with our quartic formula by substituting $k+1$:

$$\begin{aligned}
C(k+1) &= \frac{1}{24}(k+1)^4 - \frac{1}{4}(k+1)^3 + \frac{23}{24}(k+1)^2 - \frac{3}{4}(k+1) + 1 \\
&= \frac{k^4 - 2k^3 + 11k^2 + 14k + 24}{24}
\end{aligned}$$

Since both calculations for $C(k+1)$ yield the same result, we conclude and have proven that

$$C(k) = \frac{1}{24}k^4 - \frac{1}{4}k^3 + \frac{23}{24}k^2 - \frac{3}{4}k + 1.$$

CLOSING REMARKS

Yes! Our formula is correct! However, more important than the formula itself was how we found and proved it. Our first conjectured formula, although simple and elegant, did not quite work; our second conjectured formula, a quartic equation, was much more complicated, but did the job. Moral of the story? Make sure to examine enough terms before using inductive reasoning.

THIS IS MATH? SOME FAVOURITE PUZZLES FOR MIDDLE AND HIGH SCHOOL

BY SUSAN MILNER, UNIVERSITY OF THE FRASER VALLEY

Parts of this discussion have appeared in one or another of my vignettes for the Canadian Mathematics Educators Forum (CMEF) 2014. For more discussion of the mathematical benefits of doing puzzles, see Canadian Mathematical Society Notes (Vol.45.5 p 1-13).

Mathematical/logical puzzles and games appeal to a wide range of people of all ages, backgrounds and perceived strengths. In the classroom they draw students who are not normally interested in mathematics, they give students who may not excel at algorithmic manipulation a chance to show how well they can think logically to solve difficult problems and they provide interesting challenges for students who find the curriculum straightforward. Solving such puzzles encourages all students to develop mathematical habits of mind such as tenacity, attention to detail, confirming that a solution is correct, tracking down errors, being willing to start over and so forth.

In the past two years I have had the great joy of sharing mathematical/logical puzzles and games with over 4400 K-12 students and hundreds of teachers, either in the classroom or in workshops. Teachers have been delighted to see just how much concentration, hard work and sheer problem-solving skill their students display when they think they are getting away with "no math class today!"

PUZZLES

Here are some puzzles that are, in my experience, very successful with middle and high school students. There are many more available on-line and in print, but these seem to be attractive and easily accessible for most students. Once someone gets hooked on this type of mental challenge, I believe they are more likely to become open to learning to solve a wide variety of harder logical puzzles.

HIDATO

Make a chain from 1 to 25, connecting the squares vertically, horizontally, or diagonally. The solution is at the end of this article.

	19	16	14	
	17		11	13
21	**1**	10		
25			7	5
23		3		

Because this is a very accessible puzzle with only one rule, it is an excellent introduction to the realm of logic puzzles. It involves spatial reasoning and the diagonal steps are what makes the game fun. People who enjoy the simpler Hidato puzzles will find much harder ones at the official Hidato website.

RECTANGLES (SHIKAKU)

The goal is to cover the grid completely with rectangles, with no gaps and no overlaps. Each number gives the size of the rectangle that encloses it; only one number appears in any rectangle. The solution is at the end of this article.

			6	2	4	
4		6		2		
		4		4		5
2			6			
				2		2

I like this puzzle for several reasons. It combines numbers with geometry in a visceral way, making very clear the difference between prime and composite numbers and because it involves shapes, the logic required seems easily accessible to more students than that required for some other puzzles.

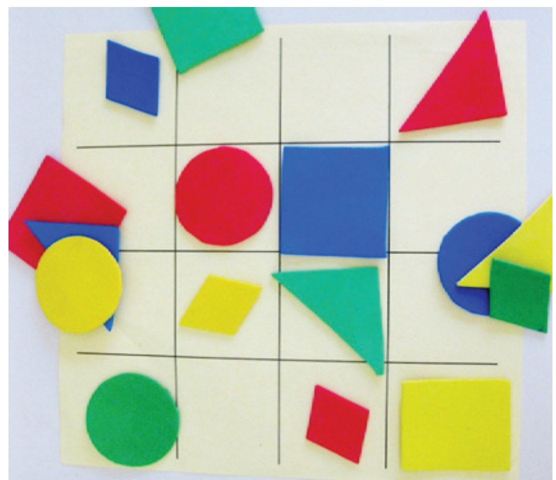
Online-versions of this come in sizes up to 19 x 19. The bigger the puzzle, of course, the larger the numbers possible, which can make for good practice with factoring. It would be fun (and instructive) for students to make their own puzzles for each other.

Students who master Rectangles might enjoy the further challenge of Filomino, which does not require that the shapes be rectangles, only that the squares involved connect along an edge.

LATIN & EULER SQUARES

This set of puzzles has many things to recommend it, among them: it is visually appealing, with bright colours and distinct shapes rather than numbers or letters; it requires puzzlers to be very clear about which rules they are using; there are thousands of solutions for each level and solutions can be found via many approaches. Also, many other puzzles become accessible once students have internalised the idea of a Latin square. Examples include Futoshiki, Kenken, Towers, Neighbours and Kakuro.

I start by asking students to sort out the bag of pieces they've been given. It's always interesting to see how many sort by colour, how many by shape and how many use both at the same time, which is essentially level 1 of the Euler puzzle below.



Unlike the way I introduce other puzzles, I usually put these rules – one level at a time – on the board, as the first students complete each level. That way each student can double-check the rules that s/he is currently using. I think it is useful for students to see the progression in complexity and to realise that the rules are what makes the game. If I don't put up the next rule quickly enough, students will often suggest it.

Getting the idea across for level 1 (below) often requires several re-statements, as some people oversimplify and others make it more complicated than necessary. I have discovered that in any age group there are people who can hear a rule and proceed to use it, others who can read it and proceed to use it, others who read a rule and need to re-state it in their own words before using it and some who need to hear the re-statement from their peers.

LATIN SQUARES

(SUDOKO PLAYERS WILL FIND THE IDEA FAMILIAR)

You'll need *n* different colours of pieces and *n* of each colour: to start with, try three different colours with three of each colour. Then try four different colours, with four of each.

LEVEL 1: Arrange the pieces so that each colour turns up only once in each row and in each column. Restatement: We want four colours in each row and in each column.

LEVEL 2: Arrange the pieces so that each colour turns up once only in each row, each column and on the two main diagonals (not always possible; it is interesting to try to figure out which sizes will allow for this).

LEVEL 3: As for level 2, but also once only on all diagonals! (also not always possible - can students figure out for which n-values this is possible?).

EULER SQUARES

(ALSO KNOWN AS GRAECO-LATIN SQUARES)

You'll need four shapes in four colours each (16 pieces) plus a 4x4 grid.

Level 1: Make every row a different colour and every column a different shape.

Level 2: No colour appears twice in a row or in a column (four colours in each row, each column).

Level 3: No colour appears twice in a row, a column or in either of the two main diagonals.

Level 4: No shape appears twice in a row, column or main diagonals (many of us find that this is harder to do than the same challenge using colours).

Level 5: No colour or shape appears twice in a row or column.

Level 6: No colour or shape appears twice in a row, column or main diagonal.

Level 7: No colour or shape appears twice in a row, column or any diagonal.

One method of solving an Euler square is provided at the end of the article.

For students who complete level 6 well ahead of the rest of the class, I make their own personal puzzle by giving them a few pieces as a starting place. It would also be possible to give a particular configuration as a starting point and ask the class to all solve the same puzzle.

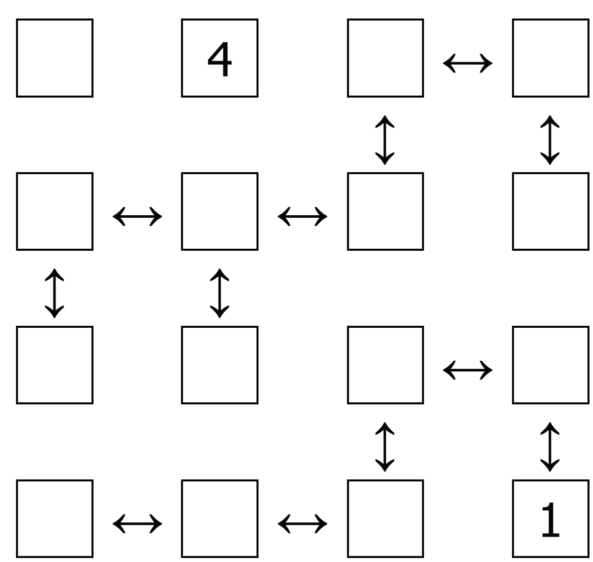
Again, you can do these puzzles with 5x5, 6x6, etc. For which sizes is level 6 not possible? Level 7?

NEIGHBOURS

Here is a popular variation of the Latin square.

Use the digits 1,2,3,4 once in each row and in each column. If there is a $\leftrightarrow$ or a $\updownarrow$ between two numbers, they are neighbours, that is, consecutive integers. For example, $2 \leftrightarrow 3$. If there is no symbol between two numbers, they are not neighbours. For example, 1 and 3 are not neighbours, nor are 2 and 4. The solution is at the end of this article.

I particularly like the "anti-clue," which is unusual.



When I introduce this puzzle to adults, there is often a sense of "aha!" in the room, in recognition of the difference in kind of clue.

Neighbours is easily accessible but can get satisfyingly difficult – Brainbashers (www.brainbashers.com) has puzzles of varying difficulties from 4x4 to 9x9.

INTRODUCING THE CLASS TO A **PUZZLE**

For nearly all types of puzzles I find it most effective to first display a completed puzzle and ask the group to figure out what the rules must be. Then we do at least one puzzle together.

I insist that students not only use mathematical language ("third row down, second column from the left") to make their suggestions, but that they justify their reasoning. I'll play dumb if I have to; it usually takes only one or two vague descriptions or incomplete arguments for students to start to be very precise.

At this point the teacher is usually grinning widely, no doubt hearing echoes of his or her own demands for clarity.

Ideally, every student should get a chance to suggest and explain a step; it can be difficult to keep the discussion from being dominated by a few quick thinkers. One teacher who regularly solves puzzles with his whole class told me that he avoids this by assigning each student a number and then using a random number generator on his calculator to choose the next student to speak. He gives his students a few minutes to exchange ideas with their neighbours before each step or group of steps, so that no one feels put on the spot. The trick of course is to keep the momentum going.

However you do it, having the class solve a puzzle or two together enables students to see that there are often several good places to start a puzzle and often several chains of reasoning to follow from any one position. Some chains are shorter and more direct and some are impressively long and hard to follow. As all (correct) chains of reasoning will ultimately lead to the same perfect satisfaction of the rules of the game, people can succeed using very different approaches. This can come as quite a surprise and often leads to many "aha!" moments along the way.

Once I think that the class has a good idea of the point of the puzzle and at least some of the logical techniques useful for solving it, I hand out a page of fairly easy puzzles. As each student finishes that page, he or she receives a page of slightly harder puzzles. Increments in difficulty should be small enough to keep frustration to a minimum but large enough to let the students know they are getting more skilled. A wise high school teacher told me that students don't start asking "What's this good for?" until they get frustrated. Students who find that they are solving introductory puzzles quickly can always be individually encouraged to skip a few -without going directly to the hardest ones, which is almost certain to induce severe frustration.

We all know that success breeds success and that there's nothing like the excitement of saying "I got that one!" to encourage students to try something slightly harder.

MORE PUZZLES

It is hard to find printed puzzles for anything other than Sudoku, but there are a number of good sources for on-line puzzles. Two of my favourites are Brainbashers and Simon Tatham's Portable Puzzles.

I have recently developed a website, which contains these and other puzzles and games at different levels. You can download and copy them for your classes or use as templates to make more of your own.
The website also contains annotated lists of on-line resources and commercial games that I have tested on people of all ages. I am still adding to the site.
There are more puzzles and games in my Dropbox; send an e-mail to susan.milner@ufv.ca if you would like access to them.

REFERENCES

Brainbashers: *www.brainbashers.com*

Canadian Mathematical Society Notes (Vol.45.5 p 1-13): *http://cms.math.ca/notes/v45/n5/Notesv45n5.pdf*

Hidato: *www.hidato.com*

Simon Tatham's Portable Puzzles: *www.chiark.greenend.org.uk/~sgtatham/puzzles*

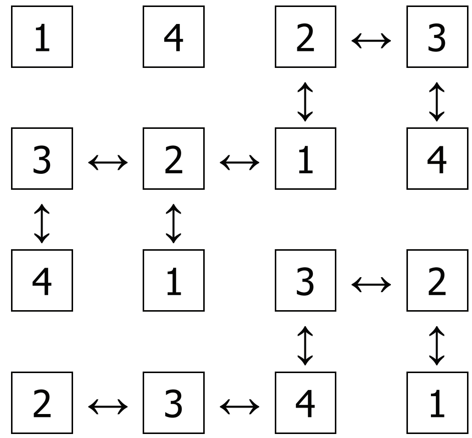
Susan's Math Games: *www.susansmathgames.ca*

SOLVED PUZZLES

HIDATO

18	19	16	14	12
20	17	15	11	13
21	**1**	10	9	8
25	22	2	7	5
23	24	3	4	6

NEIGHBOURS



LATIN & EULER SQUARES

Here is a sequence of steps that will lead to a solution of the level 3 4x4 Latin square. "Start with the diagonals" is a good hint for students who are stuck. It also works for the 4x4 Euler square.

R			B
G			Y

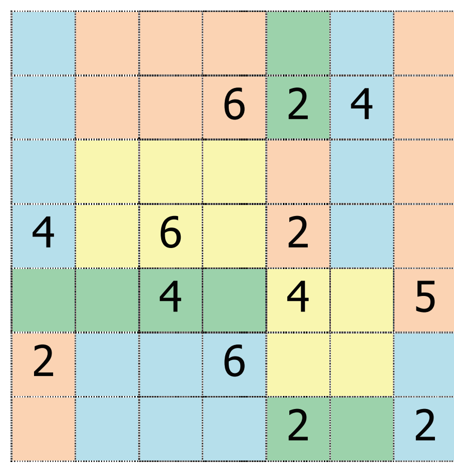
R			B
	B	R	
	Y	G	
G			Y

It is interesting to think about the number of choices we have at each stage and helpful to be aware of our choices. Note what happens if we make the opposite choice for one diagonal:

R			B
	B	Y	
	R	G	*
G			Y

There is no colour that we can place in the * square. Being aware of the choice we made at this step allows us to back up only one step in the process, rather than giving up in frustration or starting all over again.

RECTANGLES



2014 MATH CHALLENGES

SOLUTIONS WILL BE PUBLISHED IN THE NEXT ISSUE OF PI IN THE SKY.

1 Let $n \geq 1$ be a positive integer. Determine the greatest integer less than or equal to $(\sqrt{n} + \sqrt{n+1})^2$.

2 Let m, n, p be integers. Prove that 12 is a divisor of $m^2+n^2+p^2$ if and only if 12 is a divisor of $m^4 + n^4 + p^4$.

3 Let $n \geq 25$ be an integer. Find the remainder obtained when $n(n+1)(n+2)$ is divided by $n-2$.

4 The numbers 1, 2, 3, . . . , 2014 are arranged in a circle in cyclic order. We paint the numbers 1, 5, 9, and every fourth number, round and round the circle. Some of the numbers may be painted more than once. Find the number of numbers which will never be painted.

5 Find all the pairs (a, b) of positive real numbers such that

$$\frac{\sqrt{a}}{a+4} + \frac{\sqrt{b}}{b+4} \geq \frac{1}{2}.$$

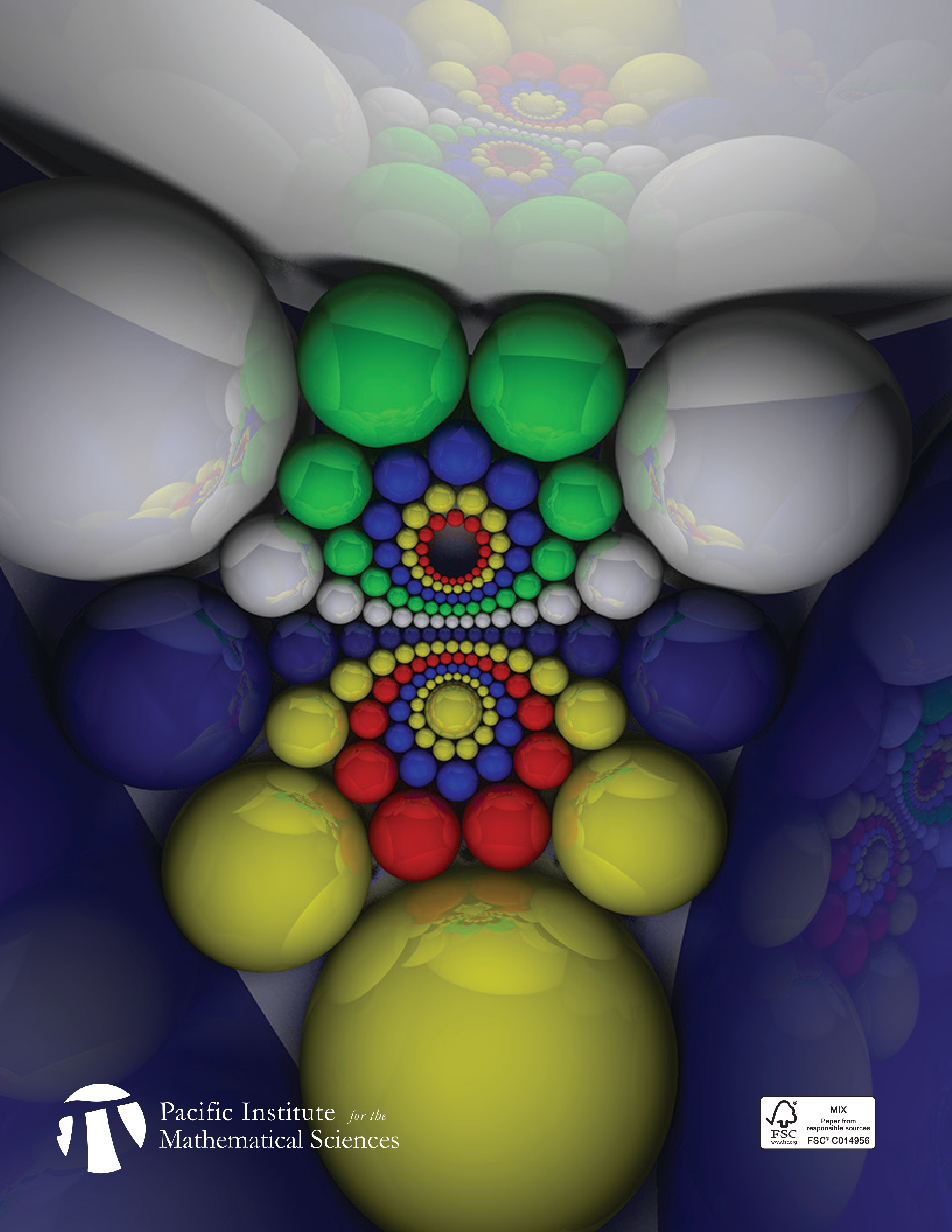
6 Find all functions $f:\mathbb{R} \to \mathbb{R}$ such that $f\left(x + \frac{3}{2}\right) \leq 2x \leq f(x) + 3$, for every $x \in \mathbb{R}$.

7 The value of a diamond is proportional to the square of its weight. A diamond breaks in two pieces and their total value is now 32% lower than the original value. Find the ratio of the weight of the larger piece to the smaller piece.

8 Find the number of isosceles acute-angled triangles with perimeter 40 such that all three sides have integral lengths.

PRIZE

PIMS IS SPONSORING A PRIZE OF $100 TO THE FIRST HIGH SCHOOL STUDENT (FROM WITHIN THE PIMS OPERATING REGION: ALBERTA, BRITISH COLUMBIA, SASKATCHEWAN, OREGON, WASHINGTON) WHO SUBMITS THE LARGEST NUMBER OF CORRECT ANSWERS BEFORE SEPTEMBER 1, 2015. PLEASE SUBMIT TO PIMS@MATH.UVIC.CA.



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