Pacific Institute for the Mathematical Sciences

Issue 16, 2012

π in the Sky

In this issue:

Mathematics of Planet Earth 2013

Pi in the Sky

Issue 16, 2012

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MATHEMATICS OF PLANET EARTH 2013

Mathematics of Planet Earth 2013 is a worldwide initiative meant to showcase and develop the fundamental role played by mathematics in a huge variety of planetary contexts. These include not only geophysical aspects connected to the structure of the planet, but also the multiple complex systems designed or impacted by humans. Climate change, sustainability, diseases and epidemics, management of resources and risk analysis are important aspects of all this. Mathematics plays a key role in these and many other processes affecting Planet Earth, both as a fundamental discipline and as an essential component of multidisciplinary research.

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THE EQUATION OF TIME

By Christiane Rousseau, Université de Montréal

In recognition of the Mathematics of Planet Earth 2013, a French version of this article has also been featured in Accromath (http://accromath.uqam.ca).

Have you ever noticed that solar noon, namely the mid-point between sunrise and sunset, is not always at the same time depending on the period of the year? The difference between solar noon and mean noon (or standard noon) is called time equation. We will see that this phenomenon has two causes, one with a period of a year and the other with a half-year period.

On the internet we can find the time of sunrise and sunset in Vancouver for all days of the year. To obtain comparable data, we will always consider Pacific Standard Time. This means we will subtract one hour from Pacific Daylight Times between the first Sunday of November and the first Sunday of March. Here are a few data:

DATE	SUNRISE	SUNSET	SOLAR NOON
January 1st	7:51	16:37	12:14
February 1st	7:31	17:17	12:25
March 1st	6:48	17:58	12:23
April 1st	5:50	18:40	12:15
May 1st	4:48	19:18	12:08
June 1st	4:25	19:53	12:09
July 1st	4:26	20:04	12:15
August 1st	4:55	19:38	12:16
September 1st	5:32	18:48	12:10
October 1st	6:10	17:50	12:00
November 1st	6:52	16:57	11:54
December 1st	7:32	16:28	12:00

The data show oscillations of 31 minutes between 11:54 on November 1st and 12:25 on February 1st, even in this limited sample. You may say that solar noon is almost always after 12:00 pm in Vancouver. This comes from the fact that Vancouver is located to the east of the center of its time zone.

We see that we have two different ways of giving the time. The true solar time (or apparent solar time) is obtained by dividing the interval between two consecutive solar noons into 24 hours. The mean solar time (the official one at a given position) is the time given by a clock that is set such that the average time of true solar noon is 12:00 pm over a year. We then introduce the quantity

solar time − standard time.

This quantity is called the *equation of time*. (In ancient astronomy, the word "equation" was used to denote a correction to be added to an averaged value to get a true value.)

We want to understand the shape of the graph of the equation of time over a year, and why this graph is nonzero. Let us first wonder why we expect solar noon to always be at the same time. We make two hypotheses:

(H1) The Earth rotates around its axis at a constant speed in the positive direction, and that its axis is oriented upwards.

(H2) The Earth rotates around the Sun at constant speed in the positive direction.

Using these hypotheses, the length of the day, namely 24 hours, should be the amount of time between two consecutive times where the Sun is at the zenith at a given point on the Earth. If we take into account the rotation of the Earth around the Sun, this length is a little more than the period of the Earth around its axis, since the Earth makes 366 rotations around its axis in 365 days.

FIRST REASON: THE ELLIPTICITY OF THE EARTH ORBIT AROUND THE SUN.

The second hypothesis is false! Indeed, by Kepler's first law, we know that the orbit of the Sun is an ellipse with the Sun at a focus. Kepler's second law says that the vector joining the Sun to the Earth sweeps out equal rays in equal interval of times (Figure 1).Hence, the closer the Earth to the Sun, the higher its speed.

In 2013, the Earth will be at the perihelion (i.e. closest to the Sun) on January 2, and at the aphelion (i.e. farthest from the Sun) on July 5. Since the Earth has its highest speed close to the perihelion, the solar days are longer than 24 hours. As a consequence, from one day to the other, the solar noon gets later and later. This phenomenon gives us the first periodic cycle of 365 days, but how can we calculate the effective time shift? We will return to this at the end when we explain how to calculate the equation of time?

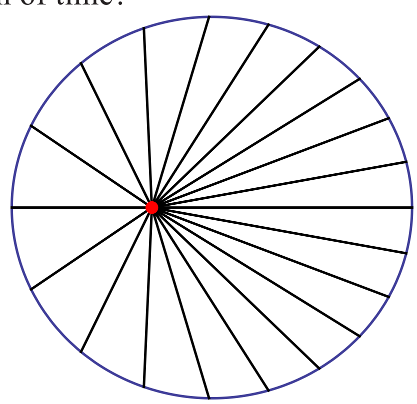


Figure 1: Kepler's second law: all sectors have equal areas. The Earth travels along the corresponding arc of the ellipse in equal times.

SECOND REASON: THE OBLIQUITY OF THE AXIS OF THE EARTH.

Modeling this second cause is more difficult! In our reasoning above, we have not taken into account the fact that the axis of the Earth makes an angle of 23.5 degrees with the ecliptic plane, namely the plane of the orbit of the Earth around the Sun. The direction of this axis is fixed during the Earth's revolution (see Figure 2).

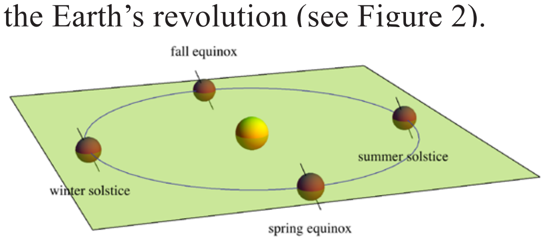


Figure 2: Revolution of the Earth around the Sun.

Depending of the period of the year, the obliquity of the axis changes the time of the solar noon. To be able to compute this effect easily, we will need to strategically choose our point of view. To simplify, we will work under the hypotheses (H1) and (H2) above.

We will choose a system of axes with origin at the center of the Earth and for which the horizontal plane is that of the Equator and the vertical axis is the axis of rotation of the Earth passing through the poles. Let us imagine that we are at the center of an immense sphere, the *celestial sphere* on which lie the Sun and the stars. Then, we see the Sun orbiting around the Earth. If the Earth's axis was not slanted, then the Sun would rotate on the equator of the celestial sphere at a constant speed. We call this movement the *mean Sun*. But since the Earth's axis is slanted, the Sun travels around a great circle of the celestial sphere in the ecliptic plane which makes an angle of 23.5 degrees with the equatorial plane, also at constant speed (see Figure 3).

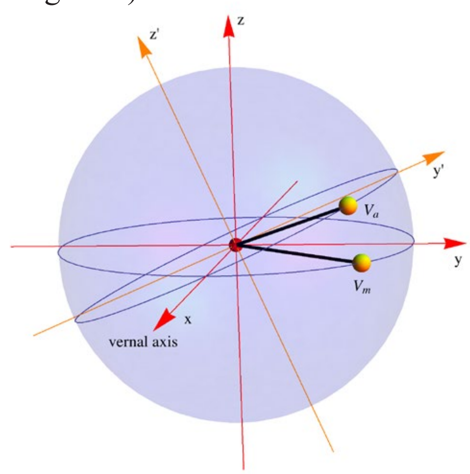


Figure 3: The two systems of axes in the geocentric point of view.

These two planes intersect along the *vernal axis*, and the intersection points of the two circles correspond to the positions of the Sun at the equinoxes. Let V_m be the vector joining the center of the sphere to the mean Sun and V_a the vector joining the center of the sphere to the true (apparent) Sun. Now, we need to introduce the fact that the Earth rotates around its axis. Let us consider the meridian half-plane trough a point of the sphere: it is a vertical half-plane containing the Earth's axis.

This half-plane cuts the Earth along a meridian. Our main observation is the following:

IT IS SOLAR NOON ALONG A MERIDIAN IF THE SUN, AND HENCE ALSO THE VECTOR V_a, ARE INCLUDED IN THE MERIDIAN HALF-PLANE.

Let us now project the vector V_a on the equatorial plane which we will observe from above (Figure 4).

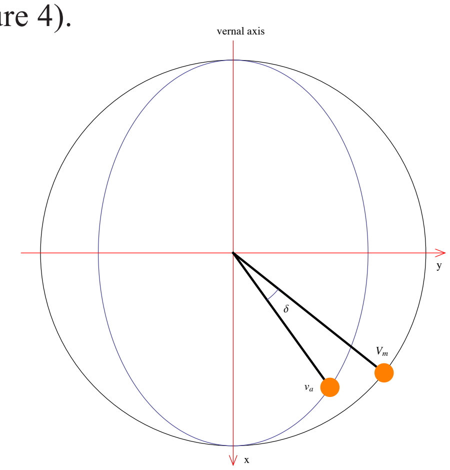


Figure 4: The projection of Figure 3 on the equatorial (horizontal) sphere.

Let v_a be the projection of V_a on that plane. This vector is also in the meridian half-plane. The angle α between the vector V_m and the vector v_a corresponds to the shift between the mean (standard) noon and the solar noon. If α is negative, since the Earth rotates in the positive direction, then a point of the Earth reaches the true noon before reaching the mean noon. Hence, the true noon is sooner than the mean noon. This is the case during spring and fall. In summer and winter α is positive and we have the contrary. The angle α vanishes four times a year, namely at the solstices and equinoxes.

What is the shift between the solar noon and the mean noon?

Since the Earth makes one rotation of 360 degrees in 24 hours, it rotates by 15 degrees per hour, which gives one degree in four minutes. Hence, the shift in minutes is four times the value of the angle in degrees. We only need to calculate α.

Computation of α

For this purpose we will use two orthonormal frames (Figure 3). In the first frame, the *x*- and *y*-axes are located in the equatorial plane. The *x*-axis is the vernal axis oriented towards the spring equinox. The *y*-axis is oriented towards the summer solstice and the *z*-axis is vertical and oriented upwards. In the second frame, the axes *x*'- and *y*'-axes are located in the ecliptic plane. The *x*'-axis is also the vernal axis. We consider the celestial sphere to be of radius 1.

The mean Sun rotates at a speed of 360 degrees per year. If the unit of time is the year, then its position at time *t* is given by

$$(x,y,z) = (\cos 360t, \sin 360t, 0).$$

By the same argument, the position of the true Sun in the second frame is given by

$$(x', y', z') = (\cos 360t, \sin 360t, 0). \qquad (1)$$

Let i', j' and k' be the unit vectors of the first frame, frame, and $\vec{i}'$, $\vec{j}$ and $\vec{k}'$ those of the second frame. We have

$$\vec{i} = \vec{i}', \quad \vec{j}' = \cos\delta\vec{j} + \sin\delta\vec{k}.$$

Also

$$V_m = \cos 360t\vec{i} + \sin 360t\vec{j},$$

and

$$v_a = \cos 360t\vec{i} + \sin 360t\cos\delta\vec{j}.$$

Let us compute the scalar product of v_a and V_m:

$$v_a \cdot V_m = (\cos 360t)^2 + (\sin 360t)^2\cos\delta.$$

We also have $v_a \cdot V_m = \cos\alpha\,|v_a||V_m|$. Hence,

$$\cos\alpha = \frac{(\cos 360t)^2 + (\sin 360t)^2\cos\delta}{\sqrt{(\cos 360t)^2 + (\sin 360t\cos\delta)^2}},$$

from which we can deduce α when taking into account its sign discussed above. The shift of time in minutes is given by four times α, where α is in degrees. Its graph is given in Figure 5.

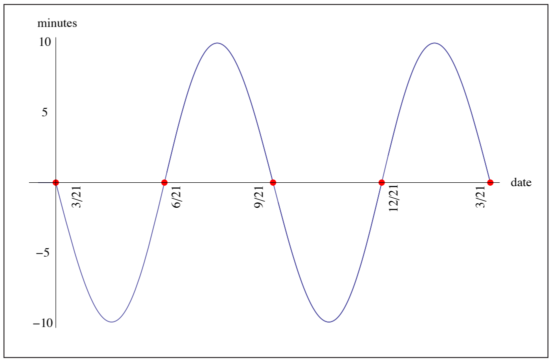


Figure 5: The graph of the equation of time for a circular orbit of the Earth, with $t = 0$ corresponding to spring equinox.

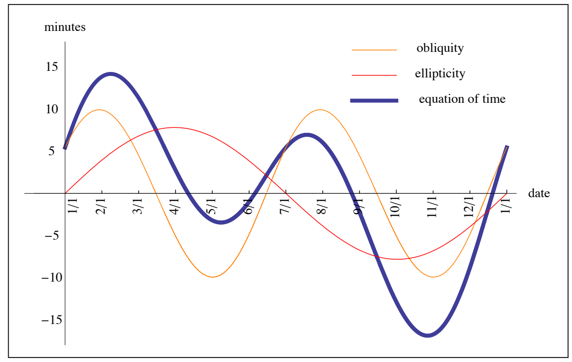


Figure 6: The graph of the equation of time over a one year period. The orange curve is the contribution of the obliquity and the red curve the contribution of the ellipticity.

HOW TO COMBINE THE TWO EFFECTS: ELLIPTICITY AND OBLIQUITY?

The combined effect is slightly different from the sum of the two shifts of time, but the sum is a good approximation of it. It suffices to modify slightly our previous reasoning on the implication of the obliquity. The only thing that changes is the equation (1). Indeed, since the speed of the sun varies, we need to replace $360t$ by the angle $\theta(t)$ of the Sun on its apparent orbit at time t. The rest of the computation is identical, but the computation of $\theta(t)$ can only be done numerically. In fact, it is easier to compute the inverse function $t = t(\theta)$. Note that the angular velocity of the Sun around the Earth in the (x', y')-plane is the same as the angular velocity of the Earth around the Sun!

COMPUTATION OF THE ANGULAR VELOCITY OF THE EARTH ALONG ITS ORBIT.

The calculation is easy if we consider the equation of the elliptic orbit in polar coordinates in a frame centered at a focus of the ellipse. Let us suppose that the half-line joining the origin to the second focus makes an angle θ_0 with the positive horizontal semi-axis and let (r, θ) be the polar coordinates of a point on the ellipse.

Then we must have

$$r = \frac{A}{1 - e\cos(\theta - \theta_0)},$$

where e is the eccentricity of the ellipse (Let us recall that if a and b are the ellipse's semi-axes and if $a > b$, then $e = \frac{a^2 - b^2}{a}$.

Let us consider a small sector swept by the vector joining the Sun to the Earth of angle $d\theta$. It is easy to be convinced that its area is approximately $\frac{r^2 d\theta}{2}$. Since $d\theta = \frac{d\theta}{dt}dt$, the area is approximately $\frac{r^2}{2}\frac{d\theta}{dt}dt$. For this area to be proportional to t, we need $\frac{r^2}{2}\frac{d\theta}{dt}$ to be constantly equal to C. Hence,

$$\frac{d\theta}{dt} = \frac{2C}{r^2} = \frac{2C}{A}(1 - e\cos(\theta - \theta_0))^2,$$

which is a differential equation with separable variables. To find its solution, we write it in the form

$$\frac{A^2}{2C}\frac{d\theta}{(1 - e\cos(\theta - \theta_0))^2} = dt,$$

and we integrate both sides. If $\theta(0) = 0$, then this yields

$$t = \int_0^\theta \frac{A^2}{2C}\frac{d\tau}{(1 - e\cos(\tau - \theta_0))^2}.$$

The eccentricity of the Earth varies slowly. Currently, we have approximately $e \simeq 0,017$. Which A and C should we take if we want the period to be one year? Since

$$\int_0^{2\pi} \frac{d\tau}{(1 - e\cos(\tau - \theta_0))^2} \simeq 6.28591,$$

we take $\frac{2C}{A^2} = 6.28591$.

Hence, we finally have the graph of the equation of time in Figure 6, which sums up the whole phenomenon over a year.

Filling up the Gasoline Tank

By Shivam Bharadwaj, C. Leon King High School, Florida & Netra Khanal, The University of Tampa, Florida

Abstract
This paper aims to solve several fundamental issues in deciding whether it is beneficial to deviate from one's path to fill gasoline for a lower price. We analyze the role played by factors such as distance, gas price, tank size and gas mileage.

Key Words
Effective price per gallon, Savings, break-even, gasoline price.

Introduction
As a result of today's economic downturn, people are trying to save as much money as possible. Gasoline has always been a hot commodity; it is often referred as "black gold" because it is versatile and can be easily transported. It is not surprising that, as the price is skyrocketing, people try to decrease their spending on this product. This is done by buying fuel efficient cars, carpooling or simply not driving at all. The United States uses about 132 billion gallons of gasoline per year, consuming, per capita, more than any other nation. Combining this ever-growing demand for gasoline and a constantly decreasing supply, results in an increase in price over time.

In this paper, we consider the following story: In an effort to save money on gasoline, Jacob alters his usual route to fill up his tank at a cheaper price. Sophia, decides to stay on path and pays the higher gasoline price. We use the variable p_1 for the price per gallon that Sophia pays at the gasoline station on her route and p_2 for the lower gasoline price per gallon that Jacob pays going out of his way. The distance in miles that Jacob travels, one way, is denoted by d. The situation is similar to the one described by the Figure 1.

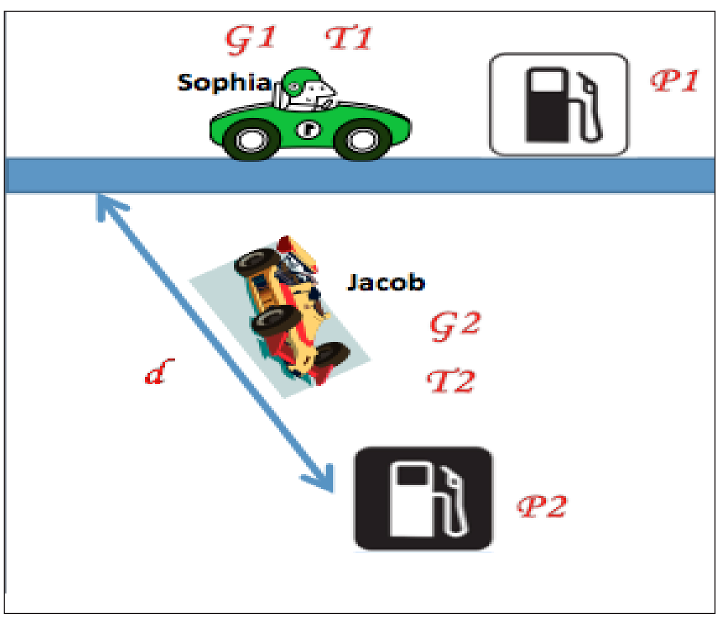


Figure 1: Pictorial description of the problem.

We also take into account the type of cars that each of them drives, along with their gasoline mileage (in miles per gallon) and tank size (in gallons). We assume that Sophia drives a car that gives G_1 miles per gallon, on average, with a tank size of T_1 gallons and Jacob has a vehicle that gives G_2 miles per gallon with a tank size of T_2 gallons. We carefully choose the types of car Jacob and Sophia drive in such a way that they provide contrast. Sports cars give less gasoline mileage in general, so it is more beneficial to look for cheaper gasoline price for sports cars than the other fuel-efficient cars.

Our goal here is to present, to the greatest extent, Jacob's benefit of travelling the extra distance, in order to better compare the two options.

This paper is organized as follows: In the next section, we discuss effective price per gallon for the type of cars they drive at some specific prices per gallon. The final section discusses several saving plans for Jacob and the maximum distance that he can travel to hunt for cheaper prices that remain economical for him.

Effective Price per Gallon

To decide whether Sophia or Jacob gets a better deal, we introduce a concept called *effective price per gallon*. The effective price per gallon, denoted by E, is given by the formula

$$E = p\frac{T}{T-S} \tag{2.1}$$

where p is the price of the gasoline per gallon, T is the tank size in gallons and S is the number of gallons one uses to go out of his or her way to fill the tank. Sophia stays on route the entire trip, so her effective price per gallon, denoted by E_1, is simply p_1. Jacob travels a distance of d, one way, to fill up his tank consuming $\frac{2d}{G_2}$ gallons of gasoline. Therefore, the effective price per gallon for Jacob is given by the equation

$$E_2 = p_2\frac{T_2}{T_2 - \frac{2d}{G_2}} \tag{2.2}$$

We assume that Sophia drives a 2012 Toyota Camry that gives 28 miles per gallon with a tank size of 17 gallons. We consider different brand of cars for Jacob, each with different gas mileages and different tank sizes. The list goes from a car giving comparatively less mileage than a Camry to one that gives comparable mileage. The reason for choosing different cars for Jacob is to analyze what price is significant for which brand of car, by traveling a given distance from the normal route. The car details given here are obtained from official company websites. The following table gives the summary of cars considered for our study:

Car Brand	Size	Mileage
Toyota Camry	17 gallons	28 miles/gallon
Corvette Coupe	18 gallons	16 miles/gallon
BMW 328i Sedan	15.8 gallons	23 miles/gallon
Ford Fusion	17.5 gallons	25 miles/gallon
Honda Accord Sedan	18.5 gallons	27 miles/gallon

Table 1: Types of vehicles considered

The gas price of \$3.50 per gallon is considered based on the price at one specific gasoline station at Tampa, Florida in June, 2012. Sophia does not go out of her way, so her effective price per gallon is \$3.50.

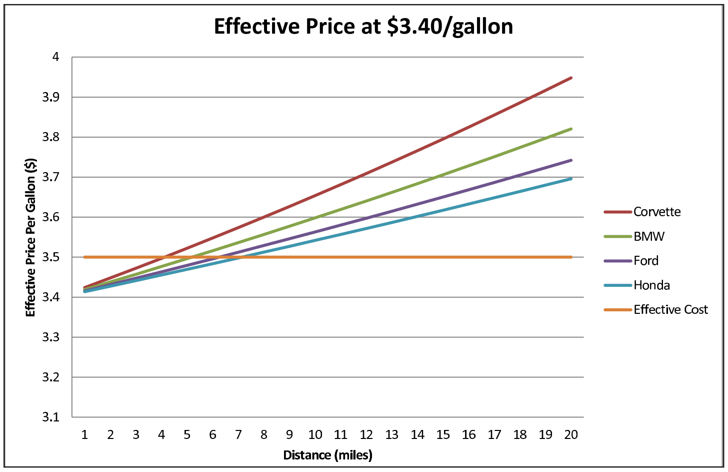


Figure 2: Effective price at $3.40 per gallon

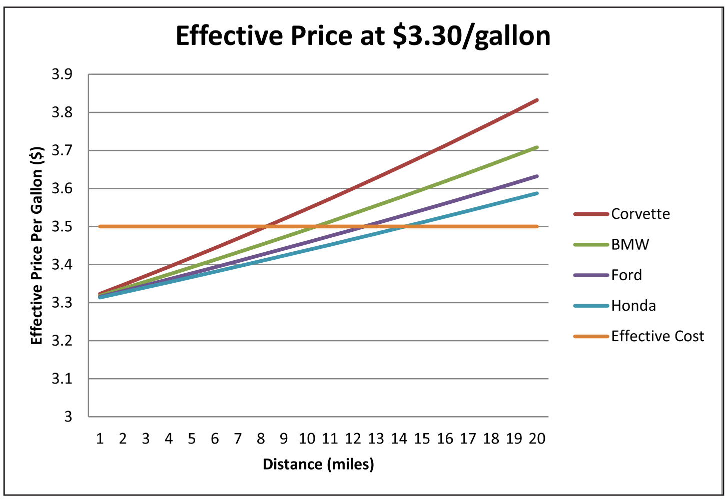


Figure 3: Effective price at $3.30 per gallon

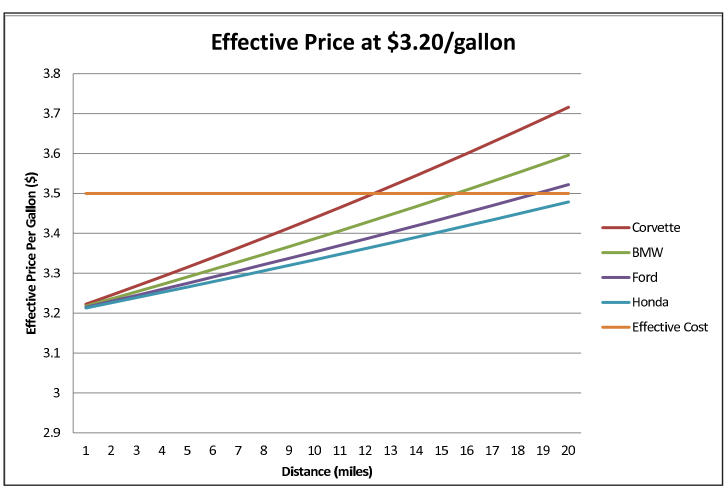


Figure 4: Effective price at $3.20 per gallon

We now test different prices that Jacob can pay and distances that he can travel for the types of vehicles considered. Distances tested vary from 1 mile to 20 miles and gas prices will vary from \$3.50 to \$3.20 at intervals of \$0.10. As a consequence, we find the results as shown in Figures 2, 3 and 4.

From the graphs, we see that as the price decreases per gallon, Jacob can drive farther to fill up his tank and still get a better deal for the type of car he drives. The horizontal red line is the effective cost if one decides not to go out of his/her way for gas. If the car that Jacob drives has results below the horizontal line at the distance to the gas station, he gets a better deal by going out of his way, whereas if it is above the horizontal, it costs Jacob more. However, a more fuel effcient car (Honda Accord in this case), is more cost-effective than any other car that Jacob opts to drive.

Break-Even Point

There is a certain break-even point at which the effective price per gallon for Sophia and Jacob is equal. For every extra mile traveled, there must be a maximum price below which Jacob gets a better deal than Sophia. The information given in Table 2 shows the maximum price (in dollars) he should consider so that it becomes beneficial to drive out of his way to fill up the tank at various distances for different vehicles.

The values presented in Table 2 are calculated by using the equation (2.2). The tank size T_2 and mileage G_2 are supplied from the data given in Table 1. The effective price per gallon E_2 is considered as \$3.499 and we calculate the price p_2 per gallon producing this E_2 -value, which will be the point at which the trip for Jacob becomes worthwhile. The distance varies for each experiment to get the price p_2 documented in Table 2.

d	Honda	Ford	BMW	Corvette
1	3.49	3.48	3.48	3.48
2	3.47	3.47	3.46	3.45
3	3.46	3.45	3.44	3.43
4	3.44	3.44	3.42	3.4
5	3.43	3.42	3.4	3.38
6	3.42	3.4	3.38	3.35
7	3.4	3.39	3.37	3.33
8	3.39	3.37	3.35	3.31
9	3.37	3.36	3.33	3.28
10	3.36	3.34	3.31	3.26
11	3.35	3.32	3.29	3.23
12	3.33	3.31	3.27	3.21
13	3.32	3.29	3.25	3.18
14	3.3	3.28	3.23	3.16
15	3.29	3.26	3.21	3.14
16	3.28	3.24	3.19	3.11
17	3.26	3.23	3.17	3.09
18	3.25	3.21	3.15	3.06
19	3.23	3.20	3.13	3.04
20	3.22	3.18	3.11	3.01

Table 2: Effective prices for break-even

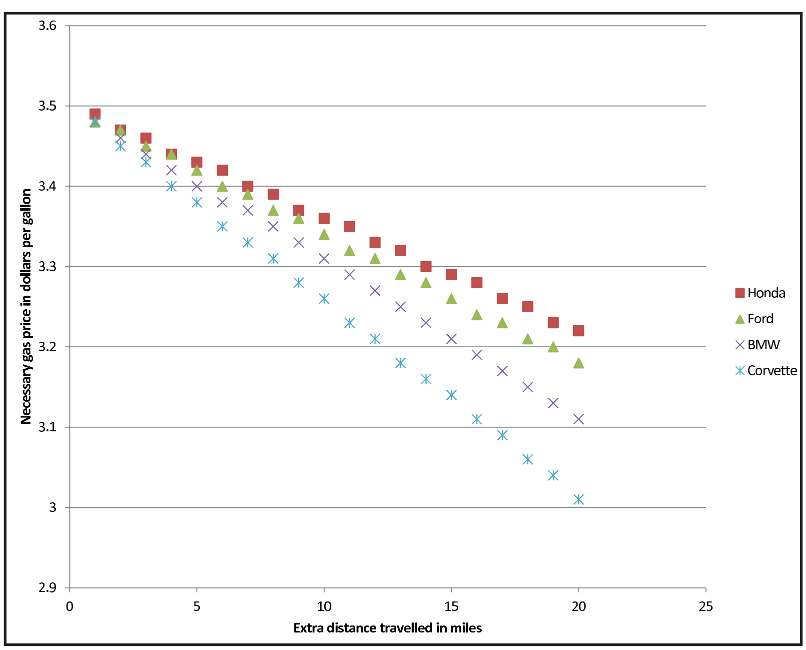


Figure 5: Scatter Diagram between the price and distance at the break-even

For the break-even point, the relationship between the price of the gasoline and the distance that Jacob can travel is shown by the scatter diagram given in Figure 5. The relationship is linear with a correlation coefficient $R^2 = 0.9996$. This shows that for every extra mile Jacob has to travel one-way, his option will only be more cost-effective if he decreases the price by a certain constant each time. The functional relationships between p and d for Honda, Ford, BMW and Corvette are given by $p = -0.014d + 3.5$, $p = -0.016d + 3.5$, $p = -0.0193d + 3.5$ and $p = -0.0243d + 3.5$ respectively.

Savings and Max Distance

Another factor we consider is how much Jacob wants to save and how far he needs to travel in order to be satisfied. Jacob's effective price per gallon, as a function of price and distance, is given by the equation (2.2). We can express the relationship of the price that Jacob pays in terms of effective price per gallon and distance as follows:

For Corvette Coupe:

$$p_2 = E_2\left(1 - \frac{d}{144}\right);$$

For BMW 328i Sedan:

$$p_2 = E_2\left(1 - \frac{d}{181.7}\right);$$

For Ford Fusion:

$$p_2 = E_2\left(1 - \frac{d}{218.75}\right);$$

For Honda Accord:

$$p_2 = E_2\left(1 - \frac{d}{249.75}\right).$$

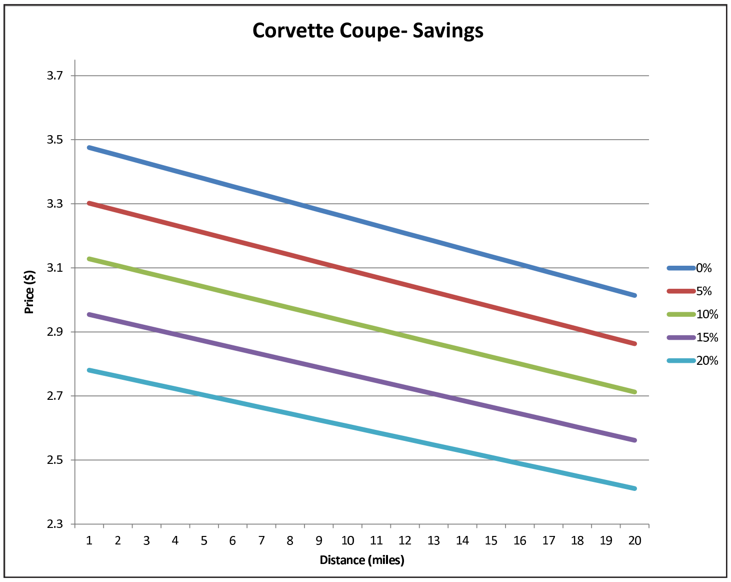


Figure 6: Saving Plans for Corvette Coupe

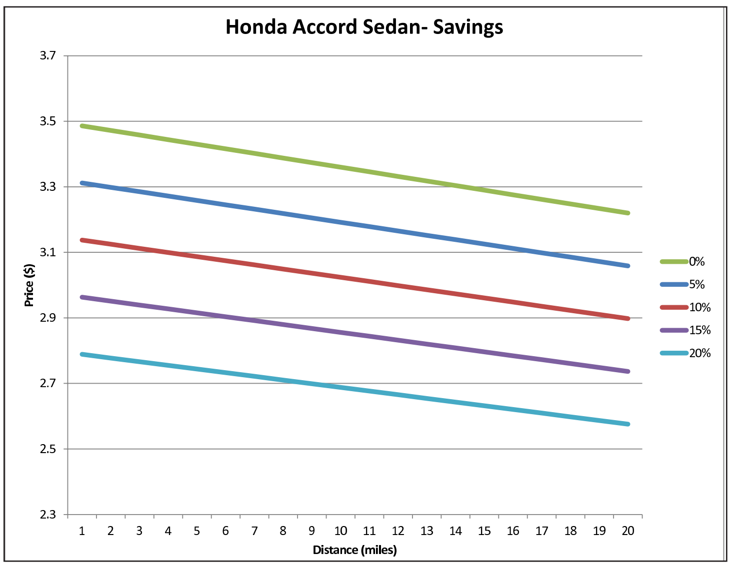


Figure 7: Saving Plans for Honda Accord

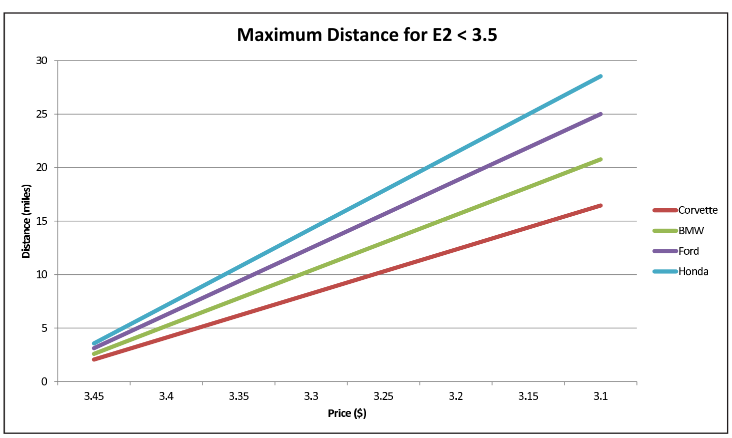


Figure 8: Maximum distance to travel for making effective price per gallon less than $3.50

We want to investigate the farthest distance that Jacob should travel to obtain different saving plans.

If Jacob desires to obtain 0%, 5%, 10%, 15%, or 20% savings, his effective prices per gallon are going to be 3.50, 3.325, 3.15, 2.975 and 2.8 respectively for each vehicle.

The graphs in Figure 6 and 7 show the relationship between the price and distance for the certain percentage of savings desired for Corvette Coupe and Honda Accord:

As of now, we have only considered situations where one variable has been fixed, effective price per gallon or price, but what if both E and p are fixed? We want the effective price per gallon for Jacob to be less than \$3.50 and p_2 to vary from \$3.45 to \$3.10. We consider equations derived before for different cars in order to find what distance d fits each of the equations; thus telling us the maximum distance at which the chosen values remain economical to travel. The following graph best describes the situation:

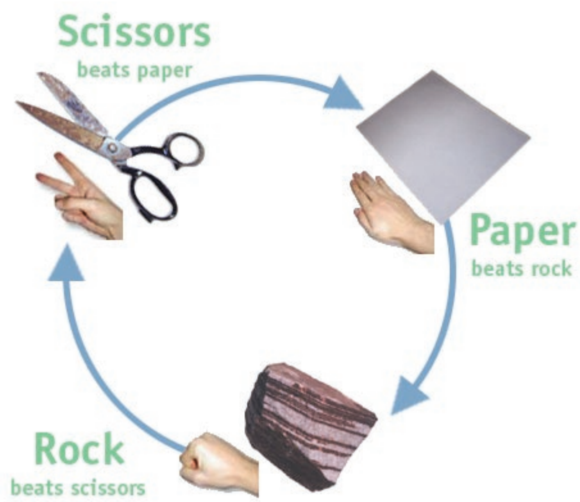
It suggests that Jacob can travel, using his BMW, around 20 miles one way if the price of the gasoline is \$3.10 per gallon and have it still remain economical to travel the extra distance. We find that with the cars that are more fuel efficient, Jacob will be able be drive a greater distance for the discounted gas price.

Rock-Paper-Scissors: A Game Theoretic Approach

By Victor Xu, Lynbrook High School, California

Introduction

Game theory is fascinating in its universal applicability. Its versatility spans economics, biology, computer science, sociology and even everyday life. Let us envision a scenario where we have all been before: engaged in a game of rock-paper-scissors.



You eyeball your opponent, taking a quick moment to ogle the last slice of pepperoni pizza before deciding on what to play. Perhaps you personally believe that people open with rock most often, and so you tend to find success playing paper on the first round. Where it gets interesting is when you play other people who share your mindset. Do you play scissors because they'll be likely to play paper? Do you play rock because you know they'll try to be thinking one step ahead? Do you play paper because they might try to psych you out by playing rock? This chain of reasoning can go on and on, and at some point you begin wondering if there is indeed a way to gain a competitive edge mathematically.

Formalization

Before I continue, let us formalize our interpretation by introducing some terminology. Classic rock-paper-scissors is a two-player, zero sum game: a game between two people in which anything that one player gains is exactly balanced by what the other player loses. For example, when two people play rock-paper-scissors, the number of rounds the first person has won is equal to the number of rounds the second person has lost. Let us measure each round through quantities of utility and the outcome of each round as a payoff where the winner gains one utility and the loser loses one utility. The payoff matrix considering all scenarios of play is displayed as follows in the form of pairs with the first coordinate being Player 1's payoff and the second being Player 2's payoff.

Player 1 \ Player 2	Rock	Paper	Scissors
Rock	(0,0)	(−1,+1)	(+1,−1)
Paper	(+1,−1)	(0,0)	(−1,+1)
Scissors	(−1,+1)	(+1,−1)	(0,0)

Table 1: Payoff matrix for both players

Let us define a strategy as a function that tells you what to do in every possible situation. For example, a strategy that states Given A, perform B and Given X, perform Y maps A to B and X to Y. Let us define a rational opponent as someone who plays according to the best available strategy at any given point in time. Finally, for the analytical purposes of this paper, assume that all games are played over an infinite number of rounds and factors of reality such as human psychology and statistical biases are unimportant. As we examine gameplay in the upcoming cases, we will be looking for strategies in which our expected utility is maximized.

Playing A Rational Opponent

Let us examine the case in which you are matched against a being who plays perfectly at every step. In other words, your opponent is not only clever enough to always figure out the strategy you are using, but will also then formulate an optimal counterstrategy in response, constantly updating throughout the game. For example, if you begin with the strategy of always playing rock, your opponent will immediately develop the strategy of always playing paper. If you then change your strategy to mirror whatever your opponent played the previous round, they will then play whatever defeats the move they made the previous round.

In essence, any deterministic strategy that takes an input and gives an output will explicitly fail against your opponent because they can determine what that output is faster than you can and adjust their play to beat it. It appears that our superhuman opponent, always able to counter our strategy cannot be defeated by regular means. Is there another strategy out there to ensure that the rational agent cannot get a competitive edge on you, regardless of his or her strategy? The answer is yes. Let p_r represent the probability of playing rock, p_p the probability of playing paper, and p_s the probability of playing scissors, the best strategy is just to play randomly with a probability distribution of $p_r = \frac{1}{3}$, $p_p = \frac{1}{3}$ and $p_s = \frac{1}{3}$. An example of how to do this would be rolling a uniformly weighted six-sided die before every round to determine your next move with 1 and 4 representing rock, 2 and 5 representing papers and 3 and 6 representing scissors. A key aspect of this is that you yourself have no idea what you will play until you play it, and if you don't know what you are going to play, neither does your superhuman opponent, thus taking away their competitive edge. In this manner, no matter how rationally superior your opponent is, he or she cannot gain any sort of advantage whatsoever if you randomly choose rock, paper, or scissors with equal probability. Intuitively, this makes a lot of sense, but let us try to find a mathematical analog for this to properly explain what is going on.

Utility Breakdown

Perhaps the best way of expressing game outcomes is through a utility function. Reverting back to the earlier payoff matrix, let us isolate the payoffs of Player 1 and approach the game from our rational opponent's point of view.

You

Opponent

	Rock	Paper	Scissors
Rock	0	−1	+1
Paper	+1	0	−1
Scissors	−1	+1	0

Table 2: Your opponent's payoff matrix

The utility function U will output your opponent's payouts of playing rock, paper, or scissors individually taking into account any of your responses. Again, let the probability that you will play rock be p_r , that you will play paper be p_p, and that your will play scissors be p_s. Clearly, $p_r + p_p + p_s = 1$. Looking at the row for which Agent Rationality plays rock, he receives a payoff of 0 with probability p_r , a payoff of -1 with probability p_p and a payoff of 1 with probability p_s. His overall utility from playing rock can then be expressed in the form

$$U(rock) = 0 \cdot p_r - 1 \cdot p_p + 1 \cdot p_s$$

Likewise, his utilities from playing paper and scissors can be expressed in a similar fashion.

$$U(paper) = 1 \cdot p_r + 0 \cdot p_p - 1 \cdot p_s$$

$$U(scissors) =- 1 \cdot p_r + 1 \cdot p_p + 0 \cdot p_s$$

Plugging in the probability distribution $p_r = \frac{1}{3}$, $p_p = \frac{1}{3}$, $p_s = \frac{1}{3}$, we find *U (rock)* = 0, *U (paper)* = 0 and *U (scissors)* = 0. It is apparent that our rational opponent's expected utility gain for playing any move, whether rock, paper, or scissors is zero, thus proving that regardless of what he does, he will be unable to gain an advantage against the random distribution $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. In this sense, we have found a "winning" strategy.

Playing A Rational Opponent II

Suppose your rational opponent is getting frustrated and wants to change the rules. Now let us make things interesting. We'll change the game so the payouts are no longer uniform but the game is still zero sum. If you play scissors when he plays paper, it is now equivalent to winning two rounds, or in other words, you gain two utility while your opponent loses two utility. Of course, your opponent, being perfectly rational, immediately finds the optimal strategy, but more importantly how does *your* strategy change? Do you play scissors more because a win is worth twice as much? Do you play paper more because your opponent may play rock more often predicting a spike in the rate you play scissors? Again, this chain of cyclical thought can continue for a frighteningly long time, but once again we can look for an answer mathematically.

As previously established, your opponent, being an embodiment of rational perfection, will immediately discern whatever strategy you choose and come up with an optimal counterstrategy, serving as the basis for why all deterministic strategies fail and why we must turn to probabilistic strategies. In classic rock-paper-scissors, the probability distribution $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ that we chose worked because our opponent's utility gain was not just zero, but zero for all of his possible moves. In this variation, the goal of our mixed strategy is to make the utilities of playing rock, paper, or scissors for your opponent equal. At this point, regardless of whether your opponent knows exactly what you will do and what your godlike opponent decides to do, he cannot improve his utility gain and thus you have achieved the goal of minimizing your losses and maximizing your payoffs. Let us express this idea with in terms of functions of utility.

		You		
		Rock	Paper	Scissors
Opponent	Rock	0	−1	+1
	Paper	+1	0	−2
	Scissors	−1	+1	0

Table 3: Your opponent's new payoff matrix

From this table we can determine your opponent's utilities for playing rock, paper, or scissors. We obtain as follows:

$$U(rock) = 0 \cdot p_r - 1 \cdot p_p + 1 \cdot p_s$$
$$U(paper) = 1 \cdot p_r + 0 \cdot p_p - 2 \cdot p_s$$
$$U(scissors) = -1 \cdot p_r + 1 \cdot p_p + 0 \cdot p_s$$

Since we want these three utility values to be the same, we can equate the three expressions to obtain a system of equations which we can solve to find our winning probability distribution.

$$-p_p + p_s = p_r - 2p_s$$
$$-p_p + p_s = -p_r + p_p$$
$$p_r + p_p + p_s = 1$$

Rearranging, we can express the system as

$$-p_r - p_p + 3p_s = 0$$
$$p_r - 2p_p + p_s = 0$$
$$p_r + p_p + p_s = 1$$

We can express this as an augmented matrix and reduce it through row operations.

$$\left[\begin{array}{ccc|c} -1 & -1 & 3 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{5}{12} \\ 0 & 1 & 0 & \frac{4}{12} \\ 0 & 0 & 1 & \frac{3}{12} \end{array}\right]$$

Finally, we find that the optimal probability distribution is $(\frac{5}{12}, \frac{4}{12}, \frac{3}{12})$. Plugging this back into our utility function, we find that $U(rock) = -\frac{1}{12}$, $U(paper) = -\frac{1}{12}$ and $U(scissors) = -\frac{1}{12}$, and so our superhuman rational opponent always loses $\frac{1}{12}$ utility regardless of whatever he plays.

Analysis

The distribution we chose was not a coincidence at all. In fact, it was a point of equilibrium, more generally known as a Nash equilibrium. Loosely speaking, a Nash equilibrium occurs when two or more players each have an optimal strategy such that neither has anything to gain by switching strategies if the others' strategies are unchanged.

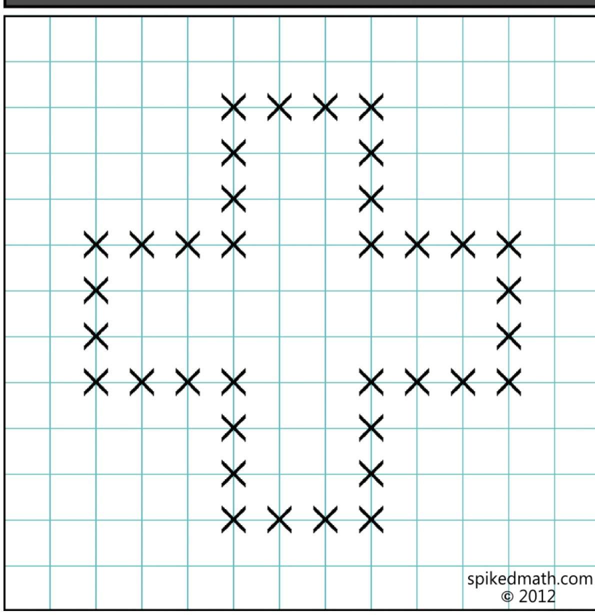
Essentially, within a Nash equilibrium, every player involved is making the best possible decision and in our case, our process led us to our winning distribution, the optimal response to our opponent's rationality. Nash equilibria constitute one of the pinnacles of modern game theory and their proposer, John Nash, was the subject of the Academy Award winning film, *A Beautiful Mind*.

Further insight into the reasoning and methods we used to arrive at our Nash equilibrium can be found through in-depth readings regarding the Minimax Theorem and the Weighted Majority Algorithm.

Closing

Phew! That was a long journey. We can now shake hands with our rational agent, call it even and both be on our separate ways. Now when I play rock-paper-scissors with my friends, even when I lose, I accept it with grace. After all, I can hold my own against any rational opponent, and now so can you!

Morpion Solitaire



The game is played on an infinite square grid. The starting configuration (see left figure) has a set of 36 points marked on the grid in the form of a cross. Each turn, you make a move by marking one additional point (where lines cross on the grid) to create a new line of 5 marked points in a row. The line cannot overlap any previous line (but can cross or touch previous lines). The goal is to create as many new lines as possible before reaching a final configuration in which no further legal moves are available.

Puzzle: What sequence of moves guarantees at least 178 consecutive legal moves?

Using mathematics to understand disease eradication: Guinea Worm Disease points the way forward

By Robert Smith?, The Department of Mathematics, The University of Ottawa

In recognition of the Mathematics of Planet Earth 2013, a French version of this article has also been featured in Accromath (http://accromath.uqam.ca).

How can we actually eradicate a disease? And why aren't we better at it? We have a very poor record of disease eradication. In the entirety of human history, we've successfully eradicated just two diseases: smallpox and rinderpest (the latter a cow disease, declared eradicated in 2011). Our "model" for what it means to eradicate a disease is thus based on what worked for these two diseases: a successful vaccine.

Guinea Worm Disease tells a different story and one that may illuminate a new way forward. Guinea Worm Disease is a parasitic disease, spread via drinking water, that has been with us since antiquity (it's mentioned in the Bible and Egyptian mummies suffered from it). Essentially, the parasite attaches itself to a water flea, you drink the flea and your stomach acid dissolves the flea, leaving the parasite free to invade your body. Because of gravity, it usually makes its way to the foot, where it lives for an entire year. **See Figure 1.**

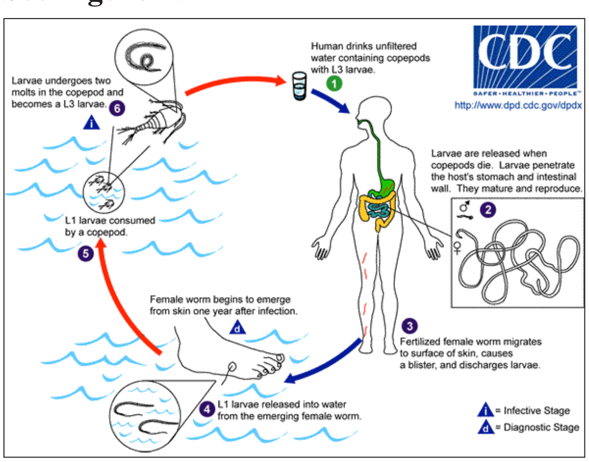


Figure 1: The life cycle of Guinea Worm disease. *Image copyright the United States Centers for Disease Control and Prevention.*

After a year, your foot is burning and itching, so you put it in the water. If your village only has one water source, then that often ends up being the drinking water. At this point, the fully grown worm bursts out of your foot, spraying forth 100,000 parasites and restarting the process. **See Figure 2.**

Figure 2: A Guinea Worm.

In the 1950s, Guinea Worm Disease affected 50 million people across most of Africa, Asia and the Middle East. Today it's on the verge of being eradicated, with less than 2000 cases, in just four African countries. Ghana was declared worm-free in 2011 and the disease primarily persists in South Sudan, as a result of the Sudanese civil war. This ancient scourge is almost gone. **See Figure 3**.

So what happened? Before we reveal the answer, let's think about how you might eradicate a water-borne disease (i.e., a disease transmitted through contaminated water).

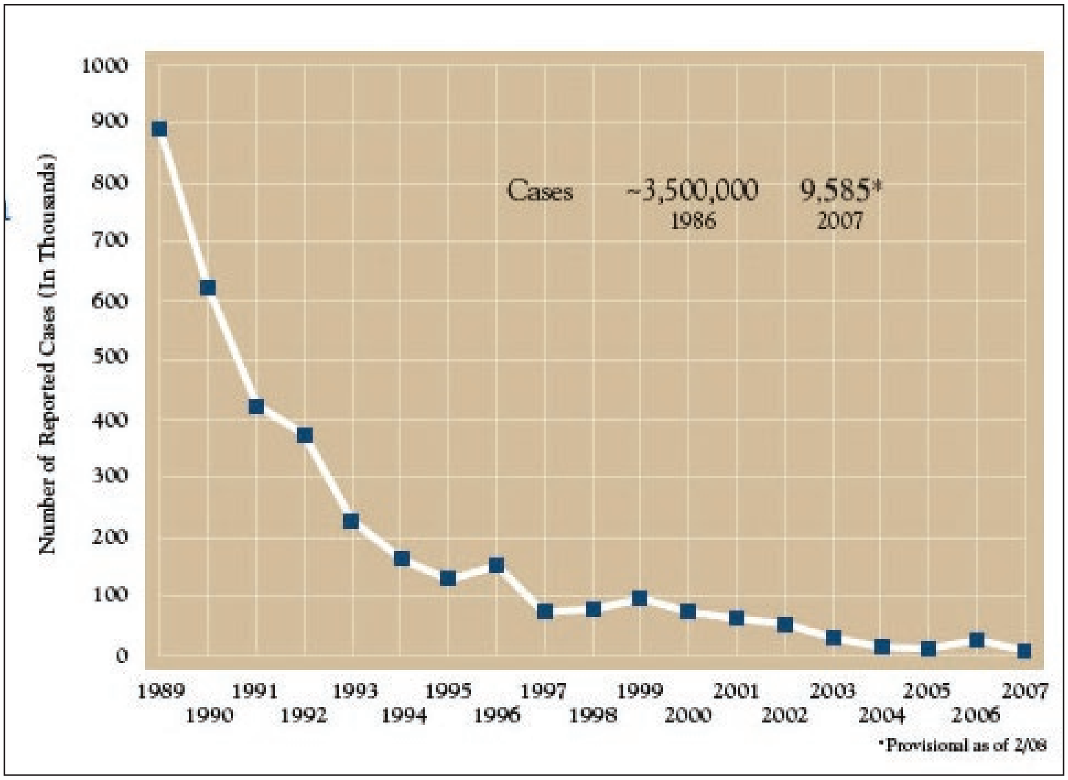


Figure 3: The decline in Guinea Worm Disease cases over the past 25 years.

Possibilities are a vaccine, drugs that treat symptoms, chemicals that kill the parasite, better hygiene or education that changes people's behaviour. Unfortunately, there is neither a drug nor a vaccine to treat Guinea Worm Disease. So let's see what mathematics tells us.

Mathematical modelling of infectious diseases is a fairly new topic that has had significant success. It has been useful in programs dealing with malaria control, smallpox eradication, mosquito management, climate change and emergency preparedness. Where modelling works well is in quantifying measurable things, like drugs, vaccines or insecticide. Where it has more trouble is with messy and unpredictable variables, like human beings.

Incorporating human behaviour into models is complex and requires an understanding of the ethical, sociological and biomedical factors inherent in tackling a disease. This requires interdisciplinary research across the traditional boundaries of social, natural and medical sciences.
To create a mathematical model, we need to keep track of what comes in and what goes out. In the case of Guinea Worm, we divide the population of humans into three subcategories. The first category is susceptible individuals; three things can happen to them: they are born, become infected or die. The second is infected individuals, who either become infectious or die. The third, infectious individuals, either recover or die. We also have a population of worms: the parasite is born when infectious individuals put their foot in the drinking water (because fresh water produces relief) and dies shortly thereafter. Guinea Worm disease is not lethal, so each time we speak of death rate, it is the usual death rate.

Combining these, we develop a system of differential equations that describes the rates of change of every variable. This system is kind of an "engine of change." With a starting key (the initial conditions), we can then use our engine to predict the future. This procedure works if we've gotten the mechanics of the interactions right.

Modelling is like map-making.

You don't want a map to be a perfect representation of reality, because that would be too cumbersome. Instead, you want the salient features, scaled down to a usable size.

So modelling isn't trying to mimic reality, but instead it's providing a useful roadmap so you can navigate the future. **See Figure 4.**

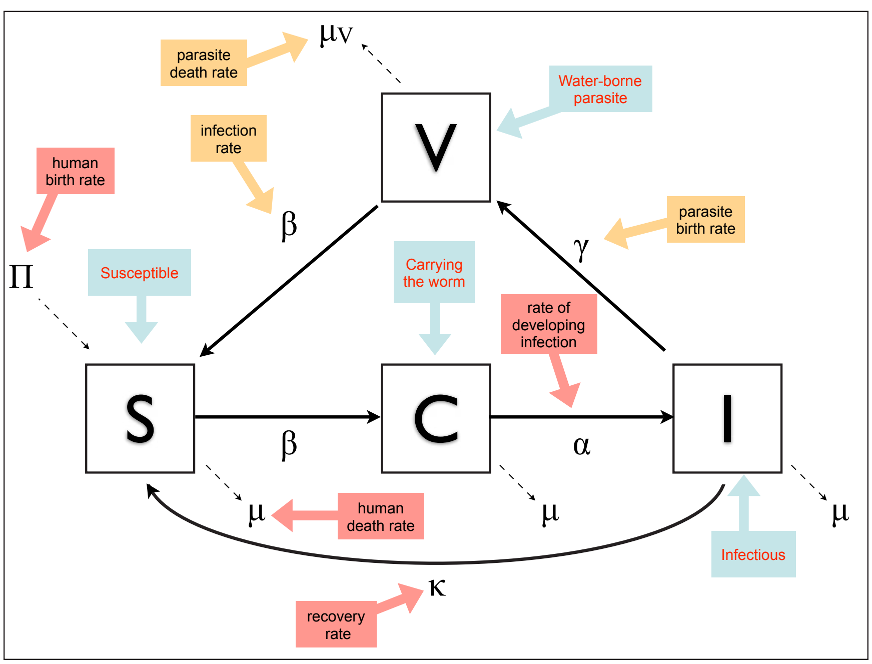
How do we know when we've eradicated a disease? Or at least when we're moving in the right direction?

This issue vexed public health officials in the early twentieth century when they were trying to eliminate malaria from places like the United States and Canada. Sir Ronald Ross won a Nobel Prize for demonstrating that malaria was spread by mosquitos (rather than toxic vapours, as was previously thought). However, this led to some despair, because it was realised that you couldn't eliminate all the mosquitos. Nor would you want to, because they prop up our ecosystems.

Ross's breakthrough came when he realised that you didn't have to kill every mosquito, but rather just a critical number of them.

This is essentially the "tipping point" of a disease: if each infected individual causes more than one infection, then the disease will spread.

Figure 4: The mathematical model. Humans can be born, become infected or die (from background reasons, not from the disease). Once infected, humans carry the worm for a year, before becoming infectious (for a few hours). The parasite's birth is proportional to the number of infected individuals, while its death rate is proportional to the size of the population.



However, if each infected individual results in less than one infected individual, then the disease will eventually die out.

This concept is called R_0, the basic reproductive ratio (pronounced "R nought"). R_0 measures the average number of secondary infections that a single infectious individual will cause. So if each infected individual infects three people, they infect three each and so forth, meaning the disease spreads like wildfire. On the other hand, if $R_0 < 1$ (so that ten infected people infect nine, those nine infect eight and so on), then the disease will die out on its own.

If we can estimate R_0 from our mathematical model and then determine which parameters will reduce it below one, then our job is done. With those control measures in place, the disease will eventually be eradicated. R_0 helps us understand which control measures will be helpful and how intensely they should be applied.

In our case, the basic reproductive ratio is

$$R_0 = \frac{\Pi \alpha \gamma \beta}{\mu(\alpha+\mu)(\varkappa+\mu)\mu_V}$$

We have three factors under our control: increasing education (which will reduce the parasite birth rate γ), reducing transmission (which will reduce β) and chlorination (which will increase the parasite death rate μ_V). You can see how R_0 depends on all these factors. So applying any one of them should reduce R_0.

That isn't the end of the story, however. Although we have identified the beneficial factors under our control, we don't necessarily achieve eradication. And every parameter will vary, in practice, because some worms will give birth to more parasites than others or some people will be more likely to be infected.

So we need to account for variations in our parameters. Fortunately, determining parameter ranges is much easier than pinpointing a precise value. The three parameters under our control are γ, β and μ_V, so let's vary these over large ranges while fixing all other parameters at their average values. **See Figure 5.**

Killing the parasite isn't terribly effective. Why? Increasing the parasite death rate involves moving along the μ_V axis to the rear left. But the level surface is very shallow, so you need to move a long way to the back corner to get under the surface. Reducing transmissibility involves moving down the β axis. But this is on a log scale, so that takes much longer than it first appears. However, see how steep the surface is for small γ? This makes it very easy to move under it by a small change in γ. This suggests that eradication should occur if we stick to one strategy: reducing the parasite birth rate.

How can we do that?

Through education, of course!
Encouraging people not to put their infected limbs in the drinking water means that each time a worm doesn't burst into the water, that's 100,000 parasites that aren't released.
This means that, in the final push to eradication, we should concentrate our efforts on reaching remote communities, informing them about the specifics of Guinea Worm Disease and its transmission cycle.

In summary, eradicating a disease isn't just a matter of sitting around and waiting for someone to invent a vaccine. We have vaccines for less than 2% of all diseases. Both drugs and vaccines are beholden to scientific breakthroughs that consume millions of dollars but may never happen. However, education is relatively cheap, highly effective when done right and can begin immediately.

The critical element of this is getting education right. Done badly, it can look to developing countries as though the West is telling them what to do (e.g., people often reject messages about safer sex due to histories of population control).

However, culturally specific education, carefully targeted towards its audience, has the potential to change entire societies, as it has with Guinea Worm Disease.

Mathematical modelling can help us determine what needs to be done in advance and to determine which factors will have the greatest impact on the outcome. We are close to eradicating Guinea Worm Disease, one of humanity's oldest diseases, thanks to behaviour changes and education alone. Once Guinea Worm Disease is eradicated, its lessons will apply to other diseases where education can be effective, not least of which is HIV. Messages need to be carefully positioned and targeted, but if done right they have the potential to do what no amount of treatment has managed: turn a global epidemic around, using the power of education.

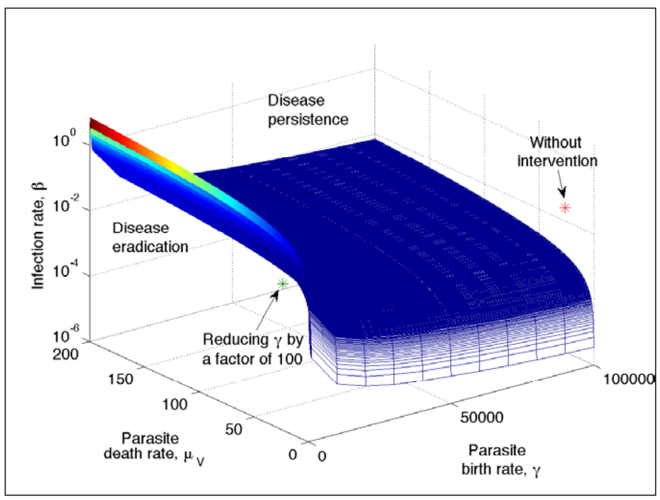


Figure 5: The level surface $R_0(\gamma, \beta, \mu_V) = 1$. If you are above the surface, then R_0 is greater than 1 and the disease will persist. If you are below, then R_0 is less than 1 and the disease will be eradicated.

HOW DO WE KNOW THAT HUMANS ARE CAUSING GLOBAL WARMING?

By Nathan Gillett, Research Scientist, Canadian Centre for Climate Modelling and Analysis

INTRODUCTION

The most authoritative reviews of climate change are carried out by a United Nations panel called the Intergovernmental Panel on Climate Change (IPCC). The most recent IPCC assessment concluded that "Most of the observed increase in global average temperatures since the mid-20th century is very likely due to the observed increase in anthropogenic greenhouse gas concentrations" (Solomon et al., 2007). But where does this statement come from? Is it based only on the opinions of climate scientists? In fact, it is based on analysis, drawing strongly on mathematics and the physical sciences.

OBSERVATIONS AND CLIMATE MODELS

Determining the causes of global warming first requires good temperature observations. Climate scientists take care to select only observations from sites which have not been subject to large local changes, for example from urbanisation around the site, and then construct datasets of temperature measurements from around the world. Such careful analysis led to the conclusion in the last IPCC assessment that global warming is unequivocal. However, this does not tell us what is driving the warming. Ideally we would run an experiment on a second Earth on which everything is kept the same, except there would be no emissions of greenhouse gases or other human influence on climate, and then compare the evolution of climate with that observed here. Since this is not possible, instead we need physically-based predictions of how climate would be expected to change in response to, and in the absence of, human influence.

In Issue 14 of *Pi in the Sky*, Adam Monahan described how mathematics and physics can be used to construct physically-based climate models (Monahan, 2010). Such models represent, in mathematical form, the major components of the climate system including the atmosphere and ocean, and the physical laws that govern their evolution. Such models aim to simulate realistic month-to-month and year-to-year climate variability in the absence of changes in greenhouse gases or other drivers (called internal variability). They can also be used to simulate climate evolution since pre-industrial times in response to the evolution of a range of climate drivers. Besides greenhouse gases, the other most important drivers are changes in particles in the atmosphere known as aerosols, which primarily act to increase reflectivity, thereby reflecting more solar radiation back to space and causing cooling. Large volcanic eruptions inject aerosols high into the stratosphere, causing a cooling which lasts several years. Lastly, changes in the brightness of the sun, including with the 11-year sunspot cycle, also influence climate.

DETECTION AND ATTRIBUTION

How do we pull all these pieces together to determine the causes of the observed changes? The process is called detection and attribution and is done using a statistical model to relate simulated and observed variations in climate. Typically we wish to avoid the assumption that the climate model simulates the correct magnitude of the response to the various climate drivers and we represent the observations using a regression model such as:

$$y = \beta_{\mathrm{GHG}} x_{\mathrm{GHG}} + \beta_{\mathrm{AER}} x_{\mathrm{AER}} + \beta_{\mathrm{NAT}} x_{\mathrm{NAT}} + u$$

Here, y is a vector of observed temperatures from which the mean has been subtracted. For example, if we had a vector consisting of observations of the global mean temperature for each decade of the last 150 years, y would be the vector formed by subtracting the 150 year mean from each of the observations - see the black line in Figure 1. In this case, y is a vector with fifteen elements, each of which is a decadal mean temperature anomaly. x_{GHG} is a similar vector of temperature anomalies from a climate model simulation of the response to greenhouse gas changes only (red line in Figure 1); $x_{\mathrm{AER}}$is the simulated response to aerosol changes only (green line in Figure 1); and x_{NAT} is the simulated response to volcanoes and solar changes only (blue line in Figure 1). Even if our climate model were perfect it would not simulate the same day-to-day weather variations and year-to-year internal variations in climate as was actually observed and therefore, we use u to represent this internal variability. The regression coefficients β_{GHG}, β_{AER} and β_{NAT} in the above equation are scaling factors applied to the simulated responses to greenhouse gases, aerosols and natural drivers respectively, which we will estimate in order to give the best fit to the observations. This will allow us to assess what proportion of the observed temperature change is due to each of these factors. We can also write this in matrix form where the columns of X are x_{GHG}, x_{AER} and x_{NAT} and $\beta = [\beta_{\mathrm{GHG}}, \beta_{\mathrm{AER}}, \beta_{\mathrm{NAT}}]^T$.

$$y = X\beta + u$$

We can estimate the regression coefficients in the above regression equation using least squares regression:

$$\tilde{\beta} = (\mathrm{X}^{\mathrm{T}}\mathrm{X})^{-1}\mathrm{X}^{\mathrm{T}}y$$

This is a multi-dimensional version of the equation that can be used to fit a straight line through points on a graph; it is the solution which minimises the sum of the squares of the vertical distances between the points and the fitted plane. Figure 2 shows the regression coefficients, $\tilde{\beta}$, derived by applying this equation to the data shown in Figure 1.

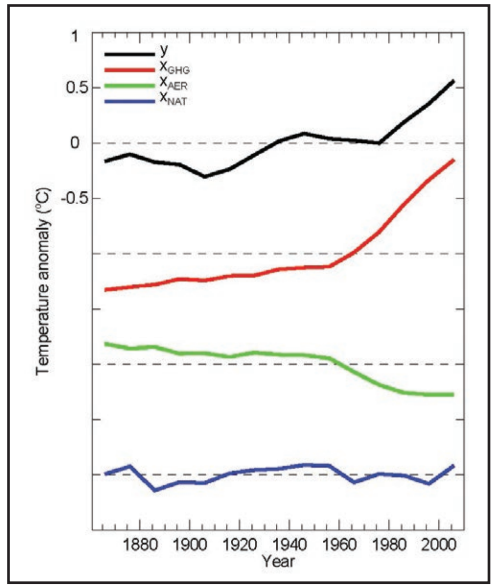


Fig. 1: Decadal mean global mean temperature anomalies from observations (y) and the average simulated responses x_{GHG}, x_{AER} and x_{NAT} from seven state-of-the art climate models.

To estimate the uncertainties in the regression coefficients, we require an estimate of the variability of climate in the absence of changes in drivers. A few studies acquire this variability estimate from long climate records of the preindustrial climate, such as from long observational records or reconstructions from tree ring widths. Generally, the length and coverage of such records is too limited, so estimates of climate variability in the absence of forced changes are taken from long control simulations of climate models with no changes in climate drivers. These are then used to derive uncertainty ranges on each regression coefficient.

If we find that a particular regression coefficient is significantly greater than zero, this means that a significant response to the forcing concerned is present in the observations (meaning it has been detected). In the example shown in Figure 2 we detect the response to greenhouse gases, aerosols and natural drivers, since the regression coefficients of all three are significantly greater than zero. The trend in the observations y can also be compared, for example, with the trend in $\tilde{\beta}_{\mathrm{GHG}} x_{\mathrm{GHG}}$, to infer the greenhouse gas contribution to the observed trend. The statement that "Most of the observed increase in global average temperatures since the mid-20th century is very likely due to the observed increase in anthropogenic greenhouse gas concentrations" is based on the results of such analyses.

The underlying quantitative meaning is that there is a greater than nine in ten chance that the trend in $\tilde{\beta}_{\mathrm{GHG}} x_{\mathrm{GHG}}$ is greater than half the trend in y. Thus, this statement is not purely based on expert opinion, but is the result of quantitative analysis based on mathematics and physics.

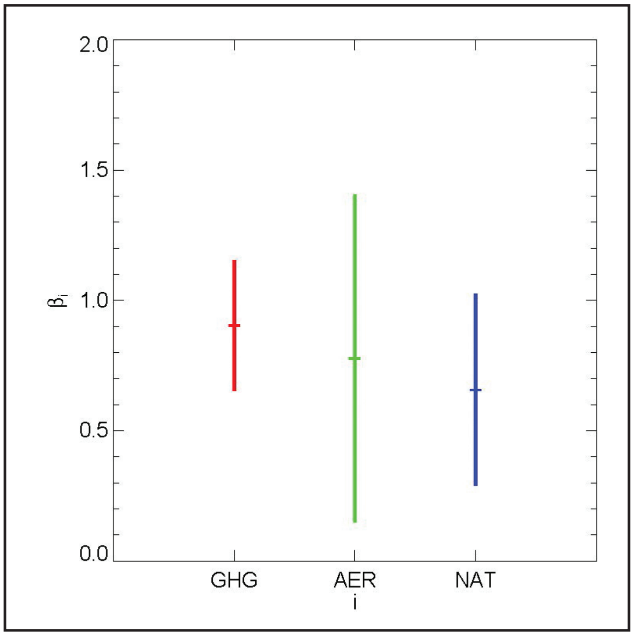


Fig. 2: Regression coefficients $\tilde{\beta}_{\mathrm{GHG}}, \tilde{\beta}_{\mathrm{AER}}, \tilde{\beta}_{\mathrm{NAT}}$ and their 5-95% confidence ranges

Various refinements to this method are applied in practice. Rather than only using global mean temperature, we typically use spatial patterns of temperature change (which helps to distinguish between them), since the pattern of temperature response to various climate drivers is different. This helps to distinguish between drivers, since their patterns of temperature response are different. The simulated response patterns do not usually come from a single simulation, but from an average of simulations of one or more different climate models. This helps to reduce the effects of internal variability and model errors.

Often signal-to-noise optimisation is used to weight the calculation of the regression coefficients towards regions with lower internal variability. And the regression equation shown above is refined to account for uncertainties in the simulated responses to each driver.

CONCLUSIONS

Detection and attribution techniques were developed in the 1990s to answer the question of whether a human effect on climate was identifiable. Using such techniques, this question has now been answered for global temperature. More recently, these techniques have been applied to many other interesting questions. For example, is there an identifiable human influence on rainfall? Has the ozone hole had a significant effect on surface climate? Is there a detectable human influence on sea ice? Have CFCs and similar compounds changed the temperature of the stratosphere? Detection and attribution studies have found that the answer to all these questions is yes, but many other questions remain to be answered.

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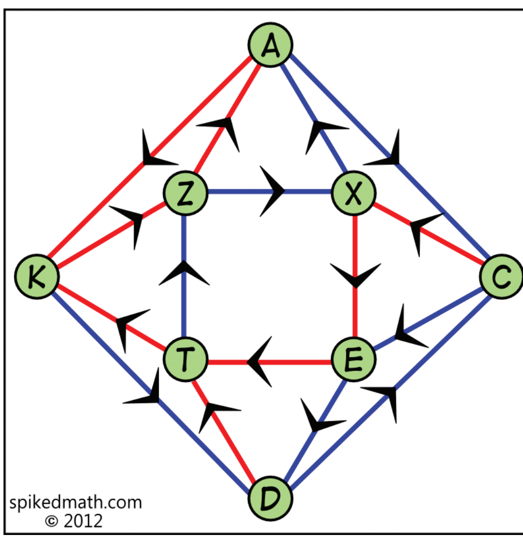

1. Pick ANY green circle and follow the path: "blue-blue-red—blue-blue-red—blue-blue-red" Write down the letter you ended at in box 1.

2. Pick ANY green circle and follow the path: "blue-red-red—blue-red-red—blue-red-red" Write down the letter you ended at in box 2.

3. Pick ANY green circle and follow the path: "red-blue-blue—red-blue-blue—red-blue-blue" Write down the letter you ended at in box 3.

4. Pick ANY green circle and follow the path: "red-red-blue—red-red-blue—red-red-blue" Write down the letter you ended at in box 4.

1	2	3	4

PI VERSUS TAU

By Rachel Hong, Leland High School, California

In the dynamic world of math, even constants are not all that constant. As my fellow math enthusiasts and I celebrated Pi Day, on March, 14, this year, I heard some calling it Tau over Two Day instead. But pi still has many loyal advocates; three months and fourteen days later, a number of them wished me a wonderful Two Pi Day instead of Tau Day.

Readers of *Pi in the Sky* might have a preference for a certain familiar circle constant, which undoubtedly makes for a more clever publication title, but these readers surely recognize the need to investigate the interesting arguments in favor of tau, which is equal to 2π. *The Tau Manifesto* and *The Pi Manifesto* detail why their respective constants are supreme, but not everyone can readily understand the discussion once it begins to delve into higher mathematics. Still, it is important to have a basic picture of any debate and understand the advantages and disadvantages of both tau and pi in the areas where they are most commonly seen.

In trigonometry the unit circle is essential, while with pi the measured value of an angle in radians can be counterintuitive. For example, the angle of a quarter of the unit circle is $\frac{\pi}{2}$ radians. But with tau, there is no unnecessary, pesky factor of two involved. A quarter of the unit circle matches $\frac{\tau}{4}$. Half of the unit circle matches $\frac{\tau}{2}$. And of course, one complete trip around the unit circle matches τ. This is easier to remember and much more elegant! That's one point for tau.

If Tau works better with the unit circle, then it must also be the more convenient constant to use with the trigonometric functions sine, cosine, cosecant and secant. This is especially evident when looking at graphs of these functions, since their period can be written as τ instead of 2π. $\frac{\tau}{2}$ is half of the period, 2τ is two periods and so on. Unfortunately, one period of the tangent and cotangent functions would have to be written as $\frac{\tau}{2}$. So, perhaps it would be the most fair to give four points to tau and two points to pi.

Young mathematicians in elementary school learn about circles and their properties well before learning trigonometry, and the most basic thing that every student soon learns by heart is how to calculate the circumference of a given circle. Using pi, the equation for finding the circumference is $C=2\pi r$. A popular argument given by the proponents of tau, also known as tauists, is that the equation is much simpler with tau: $C= \tau r$.

What about area? Area is just as important as circumference, and $A=\pi r^2$ is easier than $A = \frac{\tau r^2}{2}$. Tauists claim that tau still has the upper hand because $\frac{(quantity)^2}{2}$, or simply a quadratic expression divided by two, is commonly seen in physics. But does this make tau that much more appealing? To many mathematicians, pi is still simpler when expressing area. When it comes to the circumference and area of circles, neither tau nor pi has the upper hand and no points can be awarded.

Tau has five points, while pi has two. There is, however, one important thing that keeps tau from emerging victorious: pi has been around for thousands of years. Its digits can be found on the walls of libraries and museums all over the world. It is the first constant every mathematician learns. We have all been using it for the entirety of our mathematical journeys, however long or short they may be thus far. If those silly tauists think that they can knock out the beloved circle constant that easily, they are mistaken.

Besides, pi gives people an excuse to eat a pie with fellow mathematicians on the fourteenth of March every year and two pies on the twenty-eighth of June. This alone gives pi at least ten points!

ON THE GEOMETRY OF PHYSICAL SPACE

By Florin Diacu, Pacific Institute for the Mathematical Sciences and
Department of Mathematics and Statistics, University of Victoria

In recognition of the Mathematics of Planet Earth 2013, a French version of this article has also been featured in Accromath (http://accromath.uqam.ca).

The first person to ever wonder about the geometry of physical space was German mathematician, Karl Friedrich Gauss. Before Gauss, nobody doubted that the shortest distance between two points is a straight line. But, by 1820, he asked whether we might have to measure this distance along an arc of a circle, as we do between two points on Earth, or must heed some other path [7]. This question occurred naturally to him after discovering some unexpected geometric properties.

Gauss dealt with triangles on the sphere, which are unlike planar triangles. He looked at the distance between two points along the shortest arc of the great circle that contains them. When these points are not antipodal, such a circle is unique. Therefore three points A, B, C of the sphere that are not too far from each other can be pairwise connected by arcs of great circles (a section of a sphere containing a diameter of the sphere), thus forming a spherical triangle of sides a, b, c and angles α, β, γ (see Figure 1).

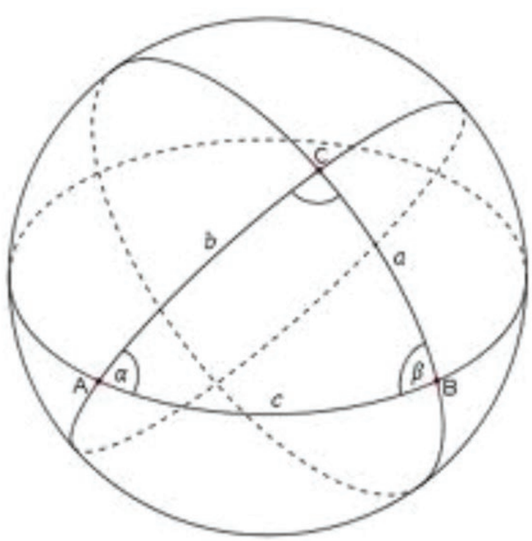

Figure 1. A spherical triangle

If you do some mental experiments with such triangles, you might rediscover a few interesting facts. Think first of a triangle with A and B on the equator and C at the North Pole. The arcs of the meridians that intersect the equator do so at right angles, so $\alpha = \beta = \pi/2$. Since γ has some positive value, this means that $\alpha + \beta + \gamma > \pi$. This fact is surprising because the sum of the angles of planar triangles is always π, but more surprises follow. Just move A and B away from each other along the equator. As the value of the angle γ increases, so the sum of the angles grows larger, which means that it is not a constant for all spherical triangles, as it is in the plane. In fact, it can be proved that the sum is always larger than π. The smaller the triangle, the closer this sum is to π, a property to be expected because a small spherical triangle is closer to being planar. There are many more interesting properties to discover, such as that the only similar triangles are congruent ones and that the area of the triangle depends on the value of the angles.

Gauss, however, had the intuition of hyperbolic geometry and understood that triangles on the sphere of imaginary radius, now called hyperbolic sphere, would have the sum of their angles always smaller than π. Moreover, he realized that the 2-dimensional sphere of Figure 1 could have a 3-dimensional analog, also called a 3-sphere and that we could similarly consider a hyperbolic 3-sphere.

Let us also consider another geometric aspect, namely curvature. Small spheres curve more than large spheres and we can express this property in terms of the radius, R, by the formula $\kappa = 1/R^2$, also due to Gauss, where κ denotes the curvature.

In other words, $\varkappa$ is the same at every point of a given sphere. But do surfaces with negative constant curvature exist? If we take a sphere of imaginary radius iR, where $i = \sqrt{-1}$, then the curvature of this object would be $\varkappa = -1/R^2$. The existence of such an object became apparent through the work of Johann Heinrich Lambert, a Swiss mathematician who preceded Gauss. These ideas, however, were not yet crystallized and it took the work of two other mathematicians, János Bolyai, a Hungarian and Nikolai Lobachevski, a Russian, who independently reached similar conclusions in the 1830s, to realize that a geometry on objects such as the sphere of imaginary radius made sense [2], [16]. Today we call this field hyperbolic geometry, as opposed to elliptic geometry, a version of which takes place on the ordinary sphere.

It is difficult to imagine these 3-dimensional geometric objects. We can understand them only through analogies and mathematical techniques that are beyond the scope of this note. Nevertheless, Gauss could not exclude the possibility that our universe is shaped like a 3-sphere or a hyperbolic 3-sphere, so he wanted to find a way to determine the geometry of the ambient space.

History is a bit murky at this point and we do not know for sure whether Gauss made his next move out of the desire to understand the physical space or just to fulfill his duties as director of the astronomical observatory in Göttingen [17], [12]. Nevertheless, in 1820 he invented the heliotrope, a new topographic instrument, and used it to measure angles of triangles between three mountain peaks near Göttingen: Inselberg, Brocken and Hoher Hagen [7]. He apparently did this in order to check whether space is hyperbolic or elliptic, in other words whether it corresponds to triangles that have the sum of the angles smaller or larger than π, respectively. His experiments failed; the limitations of his instruments provided results too close to π to allow him to draw any conclusion.

We may think that for larger triangles, Gauss's method could still provide an answer, but the scale which we would have to use for this purpose makes the idea impractical. We cannot measure the angles of triangles formed by stars for the simple reason that we cannot reach those cosmic objects. So what is the solution?

Physicists have tried to use some experiments based on the cosmic background radiation, but they have also been inconclusive [1], [19]. Moreover, they make certain physical assumptions about our universe and we cannot be entirely sure that they are correct. Nevertheless, the consensus view today is that physical space is not flat. It is unknown whether the deviation from zero is positive or negative but we do know that it must be very small.

From the mathematical point of view, however, the case $\varkappa = 0$ is highly unlikely, even if considered in an extremely small interval $(-\epsilon, \epsilon)$ of possible curvatures, with $\epsilon > 0$. The probability to hit the number 0 when throwing a point, without aiming, inside this interval is 0. According to the currently accepted cosmological theories, the fate of the universe essentially depends on whether the curvature is positive, negative, or 0, so accepting that $\varkappa$ is very close to 0 is not a satisfactory answer. But what else can we do?

Isaac Newton imagined gravitation to be the force that makes apples fall to the ground and, at the same time, keeps the Moon in its orbit around the Earth. When asking what mathematical expression this force has, he thought that it must be proportional to the product of the masses, but he could not quickly answer how this force varies with the distance. It was clear that the larger the distance, the smaller the force, but what should the exact mathematical relationship look like? To answer this question he may have thought of how the area, $\mathcal{A}$, of a sphere varies with the radius, r, namely $\mathcal{A} = 4\pi R^2$ and decided that the force should be inversely proportional to $\mathcal{A}$.

Historically, things might not have been exactly like that, since his contemporary Robert Hooke seems to have thought about this formula before him and shared it with Newton. Leaving aside the priority dispute, they agreed that the force should be of the form Gm_1m_2/r^2, where r is the distance between the bodies, m_1, m_2 are the masses and G is a constant that could be determined from the Moon's orbit.

Newton's masterpiece, *Principia*, allowed the derivation of Kepler's laws as a mathematical consequence of gravitation and gave astronomers the tools to compute the orbits of all celestial bodies. The excellent prediction of a certain comet's reappearance, made by Newton's friend Edmund Halley, established celestial mechanics at the top of the scientific pyramid [3].

But after their discovery of hyperbolic geometry, Bolyai and Lobachevsky understood that there must be a strong connection between geometry and the laws of physics and asked independently of Gauss whether the universe could be hyperbolic. Therefore they suggested the study of gravitation in hyperbolic space [2], [16]. In the spirit of Newton, they proposed a force that should be inversely proportional with the area of a hyperbolic sphere, but did not pursue the problem beyond this point. Their great achievements, which stood ahead of their time, were not recognized by their contemporaries, so Bolyai and Lobachevsky felt no boost to continue this research direction.

Soon after Bolyai and Lobachevsky died, mathematicians like Lejeune Dirichlet, Ernest Schering, Rudolf Lipschitz and Wilhelm Killing learned about these new ideas [18], [15], [13]. Schering found an analytic expression for gravitation in hyperbolic space. His thoughts were as follows: The area, $\mathcal{A}$, of a sphere of radius r inside the unit hyperbolic 3-sphere is $\mathcal{A} = 4\pi \sinh^2 r$, as geometers had already computed, where $\sinh$ denotes the sine hyperbolic function. So the force must be proportional to $1/\sinh^2 r$, where r is now the distance between the bodies.

Using a similar reasoning, Killing later proposed that, in the ordinary 3-sphere, the gravitational force is proportional to $1/\sin^2 r$. The law of masses remained the same.

Several mathematicians studied the motion of two bodies in this setting; they recovered laws similar to those of Kepler and found many other properties [14]. But recently, the study of the motion of more than two bodies began in earnest. The motivation of this research stemmed from the question we started with: what is the geometry of the physical space? If we can mathematically prove that certain orbits of celestial bodies characterize only one of the elliptic, flat or hyperbolic space, then we might be able to decide the nature of the universe by mere astronomical observations. If, say, a certain type of motion that occurs in hyperbolic space alone happens to be seen in the night sky, then space must be hyperbolic. So, the idea of measuring angles of triangles and therefore having to travel large distances to discover the geometry of the universe was replaced by the idea of sitting on Earth and making astronomical observations about how celestial bodies move.

It is first necessary to make a deep mathematical study of the dynamics in elliptic and hyperbolic space, to match the understanding of celestial motions in flat space, which has now a history of more than 300 years. But a few papers have already appeared in this research direction and we will further describe some of their exciting conclusions [4], [5], [6], [7], [8], [9], [10], [11].

In the 18th century, the French mathematician Joseph Louis Lagrange discovered that in flat space, three celestial bodies can exhibit some intriguing orbits. Assuming that their masses are m_1, m_2, m_3, then they can move as if they lie at the vertices of an equilateral triangle that rotates uniformly around its centre of mass. The distance between the bodies remains thus constant during the motion, so these orbits are suggestively called relative equilibria.

It turns out that the case of the equilateral triangle is very special, since for all the other convex regular polygons, like the square or the regular pentagon, the masses must be equal. Only for the equilateral triangle the masses can take any value. But Lagrangian relative equilibria were discovered in the solar system. The Sun, Jupiter and each of the asteroids belonging to the Trojan group form an almost equilateral triangle that rotates around its centre of mass.

To appear in nature, an orbit found through computations has to exhibit more than mathematical existence: it must be also stable, which means that nearby orbits must stay close for all time. In the case we mentioned, it means that triangles that are almost equilateral must stay close to this shape and have about the same velocities as the Lagrangian relative equilibrium. It is important to note that Lagrangian orbits are stable only if one mass is large (the Sun) and another is negligibly small (the asteroid). For comparable masses the motion is unstable, so it cannot occur in the physical space (assumed here to be flat).

Recent research, however, shows that in elliptic and hyperbolic space, Lagrangian orbits exist only if $m_1 = m_2 = m_3$. This happens because the sphere and the hyperbolic sphere have fewer symmetries than the flat space. Moreover, these orbits are unstable when the triangles are small (although they become stable for large triangles), but this can happen only at scales that are larger than the known universe, so finding such orbits in nature is practically impossible. However, since Lagrangian relative equilibria exist around us for non-equal masses, we would be tempted to conclude that space is flat. This conclusion would be a bit premature, as the Lagrangian orbits we see are not exact equilateral triangles and we don't know yet whether such orbits of non-equal masses exist in elliptic or flat space. So, at this point we have a hint, but not a proof, that space may be flat at the level of our solar system. Even if we had a proof, it would not mean much, since the solar system is like a tip of needle if compared to the rest of the universe.

However, the principle of determining the nature of the physical space through observations remains valid and we cannot exclude the possibility of finding orbits in the future, both mathematical and in real space, that can disclose the geometry of the universe. So, understanding the equations that describe the motion of celestial bodies in flat, elliptic and hyperbolic space remains an important subject of research, which will likely keep many generations of mathematicians busy. In the meantime, the answer to the original question may spring from somewhere else.

The efforts we make to understand mathematical questions breed other mathematical questions, all of which help the development of our field. Without such efforts, mathematics, science, technology, our entire culture and civilization, would not be where they are today. Working on a topic, no matter how small, finding happiness in this pursuit just because you want to learn the answer (which you may never find) is perhaps the highest level of inner freedom somebody can achieve. Only those who grasp this spirit can dedicate their lives to mathematical research.

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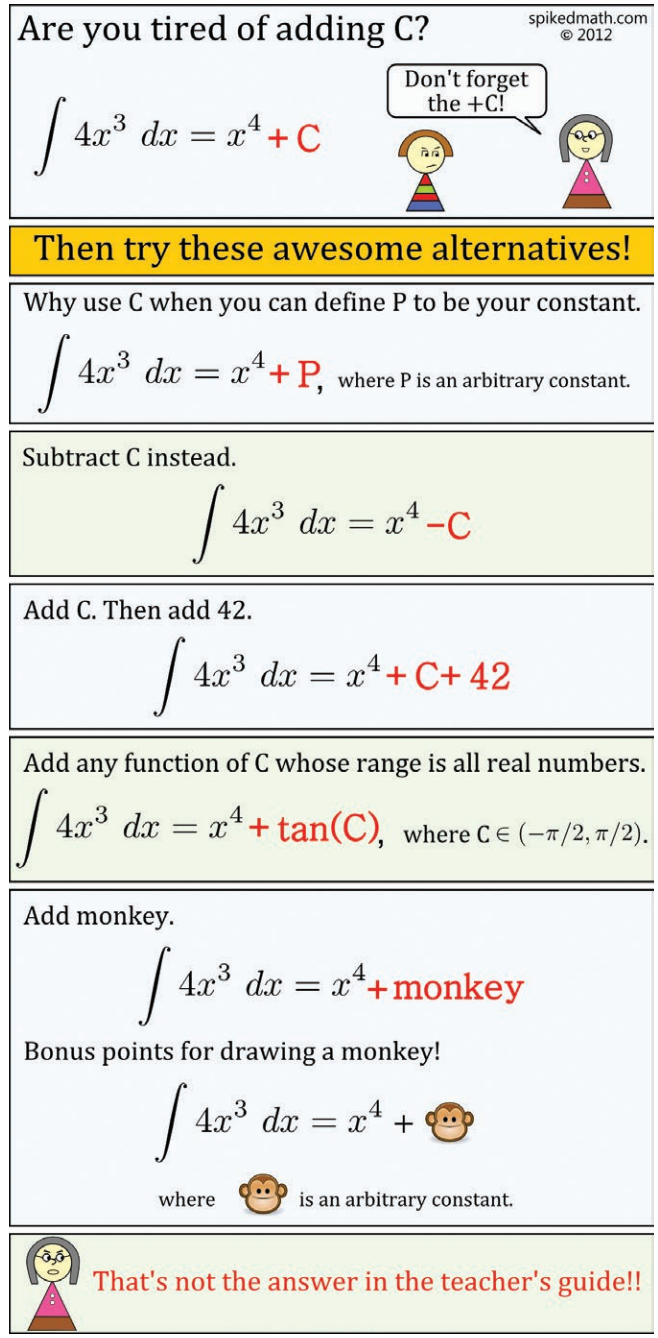

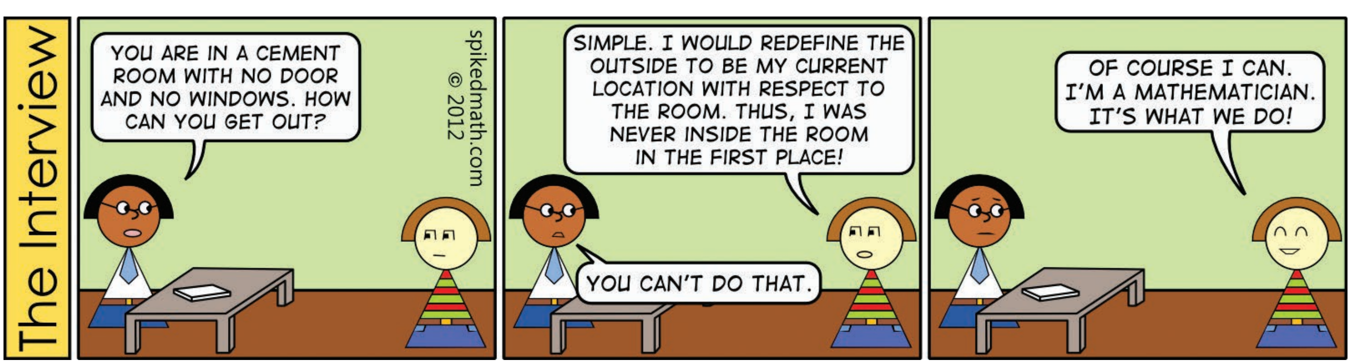

2011 MATH CHALLENGE SOLUTIONS

SOLUTIONS TO THE PROBLEMS PUBLISHED IN THE ISSUE 15 OF **PI IN THE SKY**

PROBLEM 1.

The decimal part of x = 0.2499...975 contains 2007 consecutive $9's$. Find the first 2011 decimal digits of $\sqrt{x}$.

Solution:

Since $x = \frac{1}{4} - \frac{1}{4 \cdot 10^{2009}} < \frac{1}{4}$, we may write $\sqrt{x} = \frac{1}{2} - y$, with $\frac{1}{2} > y > 0$. Then

$$\left(\frac{1}{2} - y\right)^2 = x \Longleftrightarrow \frac{1}{4} - y + y^2 = \frac{1}{4} - \frac{1}{4 \cdot 10^{2009}}$$

$$\Longleftrightarrow y^2 - y + \frac{1}{4 \cdot 10^{2009}} = 0$$

Since $y < \frac{1}{2}$, the only appropriate solution of the above quadratic equation is $y = \frac{1 - \sqrt{1-p}}{2}$ where $p = \frac{1}{10^{2009}}$.

Hence

$$\sqrt{x} = \frac{1}{2} - y = \frac{1}{2} - \frac{1}{4 \cdot 10^{2009}} - y^2 = 0.499..975 - y^2$$

where the decimal part of 0.499..975 contains 2008 consecutive 9's.

On the other hand $y^2 < \frac{p}{100} = \frac{1}{10^{2011}}$. Indeed,

$$\left(\frac{1 - \sqrt{1-p}}{2}\right)^2 < \frac{p}{100} \Leftrightarrow 1 - \sqrt{1-p} < \frac{13p}{25}$$

$$\Leftrightarrow \left(1 - \frac{13p}{25}\right)^2 < 1 - p \Leftrightarrow p < \frac{25}{169}$$

Hence the first 2011 decimal digits of $\sqrt{x}$ are 4, 9, 9, ..9, 7, 4 with 2008 consecutive $9's$.

Problem 2.

How many pairs of positive integers sum to 2011 and have a product that is a multiple of 2011?

Solution:

If (a, b) is a pair having the requested properties then $a + b = 2011$ and $ab = 2011k$, where k is a positive integer. Substituting $b = 2011 - a$ from the first equation into the second leads to the quadratic equation

$$a^2 - 2011a + 2011k = 0$$

If this equation has integral solutions then $\Delta = 2011\ (2011 - 4k)$ should be a perfect square. This is not possible since 2011 is a prime number and $2011 - 4k < 2011$.

Problem 3.

Find all the triplets (x, y, z) of real numbers such that $x + \sqrt[3]{x} = 2y$, $y + \sqrt[3]{y} = 2z$ and $z + \sqrt[3]{z} = 2x$.

Solution:

Consider the function $f(t) = \dfrac{t + \sqrt[3]{t}}{2}, t \in \mathbb{R}$ that is increasing on $\mathbb{R}$. If (x, y, z) is a solution of the given system then $f(x) = y, f(y) = z,$

We claim that $f(x) = x$. Indeed, if we assume that $f(x) < x$ then $f(f(x)) < f(x) < x$ and also $f(f(f(x))) < f(f(x)) < f(x) < x$. Now, since $f(f(f(x))) = x$ we obtain that $x < x$ which is a contradiction. Similarly, the inequality $f(x) > x$ does not hold. Thus we must have $f(x) = x \Leftrightarrow x = \sqrt[3]{x}$ and therefore $x_1 = 0, x_{2,3} = \pm 1$.The requested triplets are: $(0, 0, 0)$, $(1, 1, 1)$, $(-1. -1. -1)$.

Problem 4.

Let a be a fixed integer. Find all the functions $f{:}\,\mathbb{Z} \to \mathbb{N}$ such that for any $n \in \mathbb{Z}$

$$f(n - a) + f(n + a) \le 2f(n)$$

(Here $\mathbb{Z}$ denotes the set of all integers while $\mathbb{N}$ denotes the set of positive integers).

Solution:

The given inequality can be written as $f(n-a) - f(n) \le f(n) - f(n+a), \forall n \in \mathbb{Z}$.

If set $g(n) := f(n) - f(n+a)$ the above inequality takes the format $g(n-a) \le g(n), \forall n \in \mathbb{Z}$

from which we get $g(n-ak) \le g(n) \le g(n+ak), \forall n,k \in \mathbb{Z}$

i.e., $f(n-ak) - f(n-ak+a) \le g(n) \le f(n+ak) - f(n+ak+a)$.

Summing over k from 0 to m we obtain:

$$\sum_{k=0}^{m}(f(n-ak) - f(n-a(k-1))) \le (m+1)g(n) \le \sum_{k=0}^{m}(f(n+ak) - f(n+a(k+1)))$$

hence $f(n-am)-f(n+a)\leq(m+1)g(n)\leq f(n)-f(n+a(m+1))$

and therefore, since f takes positive values, $-\frac{1}{m+1}f(n+a)\leq g(n)\leq\frac{1}{m+1}f(n),\forall n,m\in\mathbb{Z},m\geq 0$

We conclude that $g(n)=0,\forall n\in\mathbb{Z}$, hence $f(n)=f(n+a),\forall n\in\mathbb{Z}$.
If $a=0$, it is clear that any function $f:\mathbb{Z}\to\mathbb{N}$ verifies the requested condition.
If $a=0$ then f is a periodic function of period $s => 0$. Consequently $f(n)=c_k$,
if $n\equiv k(\text{mod } s)$, $k=0, 1, \bullet\bullet\bullet, s-1$ where $c_0,c_1,\cdots,c_{s-1}\in\mathbb{N}$

Problem 5.

Given a set of $2n$ distinct points in a plane, prove that there exist n line segments joining pairs of these points such that no two of them intersect.

Solution:

There are $\frac{2n(2n-1)}{2}=n(2n-1)$ lines generated by the given $2n$ points. There exists a line Δ that is not orthogonal to any of these lines. Take a system of coordinates on Δ and let $x_1<x_2<\cdots<x_{2n}$ be coordinates of the orthogonal projections of the given points on Δ. If $A_1,A_2,\cdots,A_{2n}$ are the points corresponding to $x_1,x_2,\cdots,x_{2n}$ then $[A_1A_2],[A_3A_4],\cdots,[A_{2n-1}A_{2n}]$ are the requested n line segments.

Problem 6.

Let ΔABC be an equilateral triangle, and suppose EF is parallel to BC, where $E\in(AB)$ and $F\in(AC)$. Let O be the centroid of ΔAEF and M the midpoint of (EC). Find the angle $\widehat{OBM}$.

Solution:

Let T be the intersection of EF and BM. Since $\Delta MET\equiv\Delta MCB$ we obtain that the quadrilateral $BCTE$ is a parallelogram and consequently ΔCTF is equilateral. Also, since $\Delta OEB\equiv\Delta OFT\,(OE\equiv OF,FT\equiv EB,\widehat{OFT}=\widehat{OEB}=150^\circ$ we obtain that $OB\equiv OT$ and $\widehat{FOT}=\widehat{EOB}$.
Hence ΔBOT is isosceles and $\widehat{BOT}=\widehat{EOF}=120^\circ$. OM is median in the isosceles triangle BOT and thus it is also the line bisector of $\widehat{BOT}$, hence $\widehat{BOM}=60^\circ$ and therefore $\widehat{OBM}=30^\circ$.

2012 MATH CHALLENGES

PROBLEM 1

Find the number of primes that are less then 100 and can be written as a difference of perfect cubes.

PROBLEM 2

The sum of twenty positive integers is 462. Find the largest possible common divisor of these numbers.

PROBLEM 3

Let a, b be real numbers and $P(x) = ax^2 + 1007x + b$ such that the inequation $1 + x(x+1) + P(x)[P(x)+1] \leq xP(x)$ has at least one solution. Find $P(1)$.

PROBLEM 4

Find the values of a for which the equation

$$|x| + |x-1| + \ldots + |x-2012| = a$$

has exactly one positive integer solution.

PROBLEM 5

In ΔABC $m(\widehat{A}) = 90^\circ$ and $BC = 1$.The incircle of ΔABC is tangent to AB and AC at E and respectively F. Find the distance from the midpoint of BC to EF.

PROBLEM 6

In ΔABC, $\widehat{C}$ is obtuse and $\widehat{A} = 2\widehat{B}$ and the lengths of its sides are positive integers. Find the triangle ΔABC of the above type with minimal perimeter.

SOLUTIONS WILL BE PUBLISHED IN THE NEXT ISSUE OF PI IN THE SKY.

Pacific Institute *for the* Mathematical Sciences

An integral part of the PIMS mandate is to enrich public awareness of mathematics through outreach and to enhance mathematical training for teachers and students in K-12. PIMS is nurturing the pipeline of younger generations in Western Canada, including those with First Nations backgrounds. PIMS promotes numeracy as an integral part of development and learning.

Some of PIMS' 2012 education activities included:

Math Mania

An event for elementary and middle schools in BC that presents a variety of interactive demonstrations, puzzles, games and art designed to demonstrate fun ways of. 7 events were held with approximately 200 participants in 2012.

Elmacon

A yearly event for Grades 5 to 7 students from Lower Mainland BC and Victoria-area schools. ELMACON provides an opportunity to experience mathematics as an exciting sport. 275 students participated in 2012.

Math Central

5 million hits per month from approximately 400,000 visitors. Math Central attracts answer submissions from all over the world including Italy, Romania, Turkey and Indonesia.

Changing the Culture

A yearly one-day meeting that brings together mathematicians, mathematics educators and school teachers from all levels to improve teaching. 100 participants in 2012.

Aboriginal Scholarship Program

Since 2007 PIMS has collaborated to support scholarships for more than 57 Aboriginal students in the BC Lower Mainland.

Math on the Move

Visited 7 schools in 5 Saskatchewan school districts and delivered inquiry-based mathematics activities to 157 students. Support was provided by PIMS, the Faculties of Education and Science at URegina, and Math Central.

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