Chances are…you'll learn something new about probability

CONFERENCE NOTES

First off, let me say that I was very pleased (and honoured) to have such a great turnout for the workshop. I really enjoyed the opportunity to present, and am even more pleased to share some of the major thoughts that helped guide our conversation:

To get the workshop under way we started by working in groups and answering two closely related probability questions that had subtle distinctions in the wording. What came out of our discussion, and solving of the problems, is that often when dealing with probability, what we expect to be the answer (i.e., relying on our intuition) does not always provide correct solutions. For those involved with the BC curriculum, the infamous "Cupcake problem" comes to mind as an example of this. Keeping with the theme of doing mathematics, each group was presented with another probability problem from the high school curriculum and asked to give their answer to the group. Not surprisingly, all of the questions were answered correctly.

In solving these probability problems, a theme that emerged was attention to the subtle nuances that plague probability questions. These comments meant that we were definitely on our way to critically assessing what we know about probability; however, in order to continue our critical assessment, we needed to venture past the mathematical aspect of probability, and into the other academic fields as mentioned in the abstract. Recognition of this led the group to our first important distinction.

> **The theory of probability** has a **mathematical aspect** and a foundational or **philosophical aspect**. There is a remarkable contrast between the two. **While an almost complete consensus and agreement exists about the mathematics, there is a wide divergence of opinions about the philosophy**. With a few exceptions […] all probabilists accept the same set of axioms for the mathematical theory, so that they all agree about what are the theorems (Gillies, 2001, p.1, my bolding).

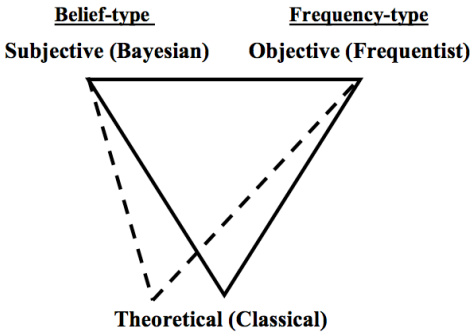
Our discussion of probability was moving from the mathematical aspect of probability to the theoretical aspect of probability, and this required yet another important distinction, which also happens to be one of the more famous quotations (in my opinion) regarding probability.

> …[P]robability…is Janus-faced. On the one side it is statistical, concerning itself with stochastic laws of chance processes. On the other side it is epistemological, dedicated to assessing reasonable degrees of belief in propositions quite devoid of statistical background (Hacking, 1975, p. 12).

In other words, the monistic perspective seen with the mathematical aspect of probability does not exist for the philosophical (or theoretical) aspect of probability.

> The idea of probability leads in two different directions: belief and frequency. Probability makes us think of the degree to which we can be confident of something uncertain, given what we know or can find out. Probability also makes us think of the relative frequency with which some outcome occurs on repeated trials on a chance setup.
> Thus far we have used both ideas [belief and frequency] almost interchangeably, because the basic rules for calculating with each are virtually identical. But now we have to distinguish between them, because the philosophical and practical uses of these two ideas are very different. **The distinction is essential […] for all clear thinking about probability** (Hacking, 2001, p.127, my bolding).

Our framework for discussion of the theoretical aspects of probability was almost complete. With Bayesians (belief-types) and frequentists (frequency-types) occupying two positions, all that was left was to situate the classical notion of probability that we had employed earlier in the workshop to answer our initial questions. According to Gillies (2001) the position of the classical notion of probability, with respect to the Bayesian and frequentist positions, is still a matter of debate for philosophers. As such, this workshop did not attempt to properly place the classical interpretation; however, all three positions were examined and the following diagram was used as a guide for our discussion of the philosophical aspect of probability.



Next we examined the frequentist interpretation of probability by having groups conduct the following experiment:

> *Put five identical, standard thumbtacks in the cup provided. Shake them, and then toss them onto the desk. Count how many of the tacks land on their flat side and how many land resting against their points. Repeat the experiment 10 times, and then use your data to calculate the probability of a tossed thumbtack landing point side down.*

Groups results were recorded on the overhead and discussion of the task focused on the differences between conducting experiments where the classical probability is

easily calculated, versus experiments were the classical probability is harder (or perhaps not able) to be calculated. The proposition put forth was that teaching the concept of frequentist probability might be better achieved through experiments with different shaped objects such as spoons, tacks and other odd shaped objects. While Gillies notion of intersubjective probability was mentioned, it was not discussed in much detail. Further examination of this point brought forth an important distinction from a broader perspective.

> The important point to be made here is that there are, indeed, different epistemological traditions in philosophy, and they have led to different research traditions in stochastics. As a result, there is not a common point of philosophical reference in the research community for doing research on stochastics and there are, consequently, opportunities for researchers to misunderstand each other (Shaughnessy, 1992, p. 468). [Furthermore, Shaughnessy notes that w]e are saddled with considerable baggage, both philosophical and historical, which can provide obstacles not only to our research in learning probability and statistics, but also our very ability to communicate results to other researchers (p. 467).

At this point we discussed how on the surface (i.e., the mathematical aspect of) probability seems to have a solid foundation. On the other hand, getting past this *Monistic Probabilistic Perspective* reveals that the philosophical aspect of probability is burdened from fundamental epistemological issues (e.g., rationalist/empiricist). Within this disarray that is probability theory, inherent subjectivity was the one common thread that was exposed through our discussion of the different interpretations of chance (e.g., assumption of equiprobability in classical, and what constitutes large in frequentist) and subsequently led to our examination of subjective probability; however, from a psychological perspective and not the philosophical perspective.

We noted there is often cavalier use of the word probability, and just because we became aware of particular theories of probability does not mean that those theories are immune from loose definitions as well. This is especially true of subjective probability. This loose definition is seen in a number of fields, which include mathematics education and psychology. Within mathematics education there exists a number of occasions where the words ‘subjective probability’ are used. However, at no point in my examination of the research has “subjective probability” been the philosophical perception of chance defined earlier, that is to say Bayesian probability. The same can be said for certain work in the field of psychology, including the work of Tversky and Kahneman as shown below.

> We use the term “subjective probability” to denote any estimate of the probability of an event, which is given by a subject, or inferred from his behavior. These estimates are not assumed to satisfy any axioms of consistency requirements. We use the term “objective probability” to denote values calculated, on the basis of stated assumptions, according to the laws of the probability calculus. **It should be evident that this terminology is**

> **noncommittal with respect to any philosophical view of probability** (Tversky & Kahneman, 1982, p. 32, my bolding).

At this point of the workshop our focus shifted back to working on probability questions. However, the questions we were working on were some of the more famous questions from the heuristics and biases literature of the 1970's. Working on these problems provided an opportunity to discuss particular heuristics (e.g., availability, representativeness, and adjustment and anchoring), and some associated biases (e.g., effectiveness of a search set, biases of imaginability; insensitivity to prior probabilities, misconceptions of chance, illusion of validity; insufficient adjustment). After looking at particular examples the group was quick to determine that "[t]he core of Kahneman and Tversky's research deals with heuristics and biases. For Kahneman and Tversky (1972), heuristics are strategies that statistically naïve people use to make **probability estimates** or in the words of the authors, **judgments under uncertainty**" (Jones & Thornton, 2005, p. 74, my bolding).

Of particular interest was our discussion surrounding bias due to retrievability of instances from what I would deem a subjective frequentist perspective. This discussion was as a perfect segue to examine the most famous of all probability problems in psychology literature, the Taxi Cab problem.

After the group was given a period of time to work on the problem they were presented with two solutions to the problem, one solved in the classical approach using tree diagrams, and the other was solved using frequency information. When each of the groups was asked which of the two solutions was a "better" solution, the room was essentially divided. This point allowed for our discussion of the frequentist hypothesis.

> …the hypothesis that some of our inductive reasoning mechanisms do embody aspects of a calculus of probability, but they are designed to take frequency information as input and produce frequencies as output (Coshmides & Tooby, 1996, p. 3). [In their work they] analyze[d] several reasons why, from an ecological and evolutionary perspective, certain classes of problem solving mechanisms in the human mind should be expected to represent probabilistic information as frequencies (p. 1).

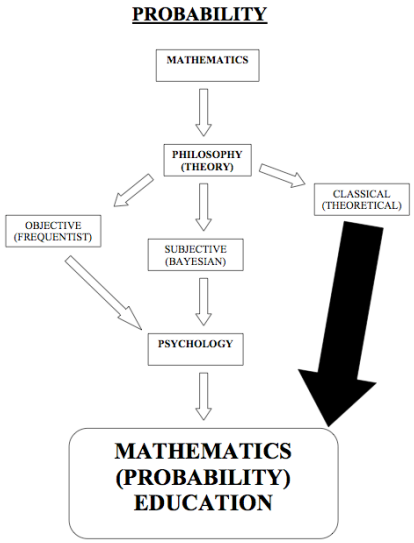
I must admit that I was surprised at the even split between the two groups. More often than not, the frequentist solution is preferred. Perhaps the discussion that had occurred just before, regarding bias due to retrievability of instances from a subjective frequentist perspective, had something to due with the outcome, but I cannot sure. Having examined probability from the perspectives of mathematics, philosophy and psychology, it was appropriate to move our discussion to the field of education. After all:

> The research of psychologists Daniel Kahneman and Amos Tversky, and many of their colleagues, has provided mathematics educators with a theoretical framework for researching learning in probability and statistics…there is little doubt of the importance of their perspective for diagnosing the psychological bases of subjects' misconceptions of probability and statistics two main types of studies are reported in the research literature.

> The first type describes how people think; the second type is concerned with influencing how people think. The first type investigates primitive conceptions of intuitions or probability and statistics, misconceptions, fallacies in thinking, judgmental biases, and so forth. The second type is concerned with influencing beliefs or conceptions, even changing them if possible. It is true that the first type has been carried out primarily by psychologists, and the second type primarily by mathematics/statistics educators (Shaughnessy, 1992, p. 470).

Colloquially put, Tversky and Kahneman are the "fathers" of probability research in mathematics education. However, "[t]he field of probability and statistics is barely a mathematical adolescent when compared to geometry or to algebra" (Shaughnessy, 1992, p. 468). Consequently, "[r]esearch into the development of probabilistic thinking and the teaching and learning of probability has occurred largely during the last 50 years" (Jones & Thornton, 2005, p. 65). A small discussion on how all of the issues talked about prior have manifested themselves in probability education being such a young field.

With that, our discussion of probability from a variety of perspectives, and how these perspectives have shaped the field of probability education, was complete, and our session was almost over. However, before the session was over I wanted to (1) summarize our probability journey (2) return to the abstract and address some embedded questions and (3) culminate our workshop on the proposition proposed at the end of the abstract. In order to summarize our probability journey I used the following diagram.



As for the abstract, the underlined portions served as five discussion points to help shape our question period.

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On Friday March 16, 2007 history was made. For the first time ever there was a three-way tie (for first place) on Jeopardy! People involved with the show decided to contact Dr. David Levine about the odds, **(#1) which he claimed were 1 in 25 million**. Interestingly, controversy over the calculation has subsequently swept the Internet. **(#2) What is all the fuss about?**

In this workshop we will critically assess what we know about probability. In order to do so, participants will solve a variety of **(#3) "classical"** probability problems (i.e., be prepared to do some mathematics). **(#4) I contend, with 83% certainty**, that by the end of this workshop, we will all have a better understanding of the Jeopardy ruckus. Why? Because answering the classical problems presented will provide a venue for discussion of major: mathematical, philosophical, psychological and educational issues inherent to probability. The discussion will culminate on the following proposition: **(#5) Our best chances for changing the culture requires *assessing*, and subsequently changing, *what we do* with probability.**

Over the past couple of years, point #5 has occupied much of my thought, and for some, the concepts of (1) *probability* and (2) *changing the culture* seem like disjoint sets. However, if examined closely enough, examples of the union of these two concepts can be found. For me, one example is a short story entitled "Report on the Barnhouse Effect" by Kurt Vonnegut, which can be found in his collection of short stories entitled *Welcome to the Monkey House*. Given the recent passing of Mr. Vonnegut, coupled with the current state of societal affairs, I decided it relevant to share that story with those who came to the workshop. I am very interested in what people thought about the story, and please do not hesitate to contact me (egan_chernoff@sfu.ca). I look forward to hearing comments and opinions from those who have since read the story.

For me, the word "probability" conjures up the image of an iceberg. While I realize that (for some) this is a tired metaphor, I feel it is an appropriate way to summarize what occurred during the workshop. According to various sources (e.g., my grade nine Science teacher), approximately one eighth of an iceberg is above water. I clearly remember him saying that examining what is above the water tells you nothing of what is below the water. Furthermore, he had the audacity to claim this fact is where the phrase "tip of the iceberg" came from. Well: Turns out he was right. I contend that this figure of speech acts as an excellent metaphor for our assessment of "probability" in the workshop. The goal of this workshop –in part and as mentioned in the abstract– was to critically assess what we know about probability. In order to do so, we solved a variety of "classical" probability problems, and our answering of these classical problems provided a venue for discussion of major mathematical, philosophical, psychological and educational issues inherent to probability. In essence, the workshop provided an opportunity for us to start to see beyond the "tip of the iceberg". Perhaps some day when we are able to discern the size and shape of the iceberg that is below the surface, we will be able to realize the idea that probability is one possible approach to changing the culture.

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