Open questions on Jacobians of curves over finite fields: *p*-ranks of curves

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Effective methods for abelian varieties
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Open questions on p-ranks of curves

Let p be a prime number. Let g be a natural number.
Let X be a curve defined over a finite field of characteristic p.

Open question B1:

If X is a generic curve of genus g and p-rank 0, what is the Newton polygon of X?

Open question B2:

What are the p-ranks of curves X which are a cyclic $\mathbb{Z}/\ell$ cover of the projective line?

Outline. Definition of p-rank;
B0: how to compute p-rank with Cartier operator;
a generic Newton polygon,
p-ranks of cyclic covers of the projective line

The p-rank

The p-rank measures the number of p-torsion points on the Jacobian or the number of roots of the L-polynomial with p-adic absolute value 1.

Fact/Def: Let X be a smooth k-curve of genus g

Then $|J_X[p](k)| = p^f$ for some integer $0 \leq f \leq g$ called the p-rank of X.

Also, $f = \dim_{\mathbb{F}_p} \mathrm{Hom}(\mu_p, J_X[p])$ where
$\mu_p \simeq \mathrm{Spec}(k[x]/(x^p - 1))$ is the kernel of Frobenius on $\mathbb{G}_m$.

Let $L(t)$ be the L-polynomial of the zeta function of an $\mathbb{F}_q$-curve X.

The p-rank of X is the length of the slope 0 portion of $\mathrm{NP}(X)$.

X is supersingular if all slopes of $\mathrm{NP}(X)$ equal $1/2$.
X supersingular implies X has p-rank 0 but converse false for $g \geq 3$.

Moduli of abelian varieties: ss versus p-rank 0

The moduli space $\mathcal{A}_g$ of p.p. abelian varieties of dimension g has dimension $\dim(\mathcal{A}_g) = g(g+1)/2$.

The p-rank 0 stratum of $\mathcal{A}_g$ is irreducible of dimension $g(g-1)/2$ (codimension g).

The supersingular locus of $\mathcal{A}_g$ has dimension $\lfloor \frac{g^2}{4} \rfloor$ (number of components is a class number).

For $g \geq 3$, the dimension of the p-rank 0 strata is strictly bigger than the dimension of the supersingular strata.

Computing the p-rank

Let C be the Cartier (semi-linear) operator on $H^0(X, \Omega^1)$.

Manin: the p-rank is $f = \dim(\mathrm{Im}(C^g))$.
Thus: one can compute f, given p, X, and a basis of $H^0(X, \Omega^1)$.

Sage: compute the Cartier matrix, Hasse-Witt matrix, p-rank and a-number for hyperelliptic curve $X : y^2 = h(x)$ with $\deg(h(x)) = 2g + 1$.

$P. < x >= PolynomialRing(GF(67))$
$X = HyperellipticCurve(x^7 + x^3 + x)$
$X.p\_rank()$
2

The algorithm is a generalization of this fact for $g = 1$:
Let $h(x)$ be separable cubic polynomial.
$E : y^2 = h(x)$ has p-rank 0 iff the coeff of x^{p-1} in $h(x)^{(p-1)/2}$ is 0.

Computing the p-rank of hyperelliptic curves

Let p odd and $h(x) \in k[x]$ degree $2g+1$ with no repeated roots.

Hyp. curve $X : y^2 = h(x)$: basis for $H^0(X, \Omega^1)$ is $\{\frac{dx}{y}, \frac{xdx}{y}, \ldots, \frac{x^{g-1}dx}{y}\}$.

Let c_r be the coefficient of x^r in the expansion of $h(x)^{(p-1)/2}$.

For $1 \leq t \leq g$, consider the $g \times g$ matrix M_t s.t. $M_t(i,j) = c_{pi-j}^{p^{t-1}}$.

Yui:

The action of the Cartier operator on $H^0(X, \Omega^1)$ wrt this basis is M_1. X is ordinary ($f = g$) if and only if $\det(M_1) \neq 0$.

The p-rank of X is $f = \mathrm{rank}(M)$ where $M = M_g M_{g-1} \cdots M_2 M_1$.

B0: Need to fix! Achter/Howe found pervasive typo in literature (Yui, Zarhin, ...) Sage computes $M = M_1 M_2 \cdots M_g$:(

Example - Hermitian curve $X : y^q + y = x^{q+1}$, $q = p^n$

The Cartier operator C acts on $H^0(X_q, \Omega^1)$.

Let $\Delta = \{(i,j) \mid i,j \in \mathbb{Z},\ i,j \geq 0,\ i+j \leq q-2\}$.
A basis for $H^0(X_q, \Omega^1)$ is $B = \{\omega_{i,j} := x^i y^j dx \mid (i,j) \in \Delta\}$.

Write $i = i_0 + p i_n^T$ and $j = j_0 + p j_n^T$ with $0 \leq i_0, j_0 \leq p-1$.

$$
\begin{aligned}
C(x^i y^j dx) &= x^{i_n^T} y^{j_n^T} C\left(x^{i_0}(x^{q+1} - y^q)^{j_0} dx\right) \\
&= x^{i_n^T} y^{j_n^T} \sum_{l=0}^{j_0} \binom{j_0}{l} (-1)^l x^{p^{n-1}(j_0 - l)} y^{p^{n-1} l} C\left(x^{i_0 + j_0 - l} dx\right).
\end{aligned}
$$

$C(x^k dx) \neq 0$ iff $k \equiv -1 \bmod p$. Need $i_0 + j_0 - \ell \equiv -1 \bmod p$.

If $i_0 + j_0 < p-1$, then $C(\omega_{i,j}) = 0$.
If $i_0 + j_0 \geq p-1$, then $C(\omega_{i,j}) = \omega_{p^{n-1}(p-1-i_0)+i_n^T,\, p^{n-1}(i_0+j_0-(p-1))+j_n^T}$.

Existence of curves with given genus and p-rank

The algorithm can be used to compute the p-rank of a fixed curve, but it is too complicated to be algebraically constructive.
Let $g \in \mathbb{N}$, $0 \leq f \leq g$ and p prime.

Let $\mathcal{M}_g^f$ (resp. $\mathcal{H}_g^f$) denote the p-rank f strata of the moduli space of (hyperelliptic) curves of genus g.

Theorem: Faber/Van der Geer

Every component of $\mathcal{M}_g^f$ has dimension $2g - 3 + f$.

Theorem: Glass/P (p odd), P/Zhu (p even)

Every component of $\mathcal{H}_g^f$ has dimension $g - 1 + f$.

There exists a smooth (hyp.) curve over $\overline{\mathbb{F}}_p$ with genus g and p-rank f.

In most cases, it is not known whether $\mathcal{M}_g^f$ and $\mathcal{H}_g^f$ are irreducible.

Open question

Let A be the generic p.p. abelian variety of dimension g and p-rank 0.

TFAE and true for A:

* the Newton polygon is $G_{1,g-1} \oplus G_{g-1,1}$ (slopes $\frac{1}{g}$ and $\frac{g-1}{g}$);
* the rank of the Cartier operator on $H^0(A,\Omega^1)$ is $g-1$,
* the a-number of A is 1.

B1 Open problem. For all p and g,

(1) are conditions * true for a generic curve of genus g and p-rank 0?
(2) does there exist a curve of genus g and p-rank 0 satisfying $*$?

(1) Yes: $g = 1,2,3$ for all p. (2) Yes: $g = 1,2,3,4$ for all p.

Existence of slopes $1/4$ and $3/4$

For all p, there exists a smooth curve of genus 4 defined over $\overline{\mathbb{F}}_p$ whose NP has slopes $1/4$ and $3/4$.

Let $\mathcal{W}$ be moduli space of p.p. abelian 4-folds with action by $\mathbb{Z}[\zeta_3]$ of signature $(3,1)$. Then $\dim(\mathcal{W}) = 3$.

Also $\mathcal{W}$ is irreducible since $\mathbb{Z}[\zeta_3]$ has class number 1.

Let S be moduli space of curves $C_f : y^3 = f(x)$ (square-free $f(x)$ degree 6). Then $\dim(S) = 3$.

The image of Torelli morphism on S is open, dense subspace of $\mathcal{W}$.

There exists a point of $\mathcal{W}$ representing abelian variety with slopes $1/4$ and $3/4$. Mantovan 2004 if p splits in $\mathbb{Q}(\zeta_3)$, Bültel/Wedhorn 06 if p inert in $\mathbb{Q}(\zeta_3)$, SAGE if $p = 3$.

Proof: inductive strategy, reduce to p-rank $f = 0$

Let ν_r be a NP type with p-rank 0 occurring in dimension r.

Let $c_r = \mathrm{codim}(\mathcal{A}_g[\nu_r], \mathcal{A}_g)$.

For $g \geq r$, let ν_g be the NP type with p-rank $g - r$ 'containing' ν_r

($\nu_g = (G_{0,1} \oplus G_{1,0})^{g-r} \oplus \nu_r$), add $g - r$ slopes of $0, 1$.

Proposition P

If there exists a component S_r of $\mathcal{M}_r[\nu_r]$ s.t. $\mathrm{codim}(S_r, \mathcal{M}_r) = c_r$,
then, for all $g \geq r$,
there exists a component S_g of $\mathcal{M}_g[\nu_g]$ s.t. $\mathrm{codim}(S_g, \mathcal{M}_g) = c_r$.

Newton polygon results for $f = g - 3$ and $f = g - 4$

Recall $\nu_{g,f} = f(G_{0,1} + G_{1,0}) + (G_{1,g-f-1} + G_{g-f-1,1})$.

Application - Achter/P. Let $g \geq 3$ and $f = g - 3$.

The generic point of any component of $\mathcal{M}_g^{g-3}$ has Newton polygon $\nu_{g,g-3}$ (slopes $0, \frac{1}{3}, \frac{2}{3}, 1$).

Application - Achter/P. Let $g \geq 4$ and $f = g - 4$.

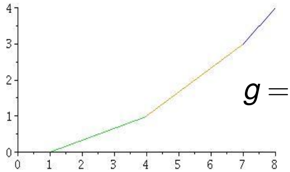
The generic point of *at least one* component of $\mathcal{M}_g^f$ has Newton polygon $\nu_{g,g-4}$ (slopes $0, \frac{1}{4}, \frac{3}{4}, 1$).

Note: When $g = 4$, there is *at most one* component of $\mathcal{M}_4^0$ whose generic NP is not $\nu_{4,0}$. If so, the NP has slopes $\frac{1}{3}, \frac{1}{2}, \frac{2}{3}$.

A generic Newton polygon

If $f = g$, the NP is $g(G_{0,1} + G_{1,0})$ (slopes $0, 1$).
If $f = g - 1$, the NP is $(g-1)(G_{0,1} + G_{1,0}) + G_{1,1}$ (slopes $0, \frac{1}{2}, 1$).
If $f = g - 2$, the NP is $(g-2)(G_{0,1} + G_{1,0}) + 2G_{1,1}$ (slopes $0, \frac{1}{2}, 1$)

If $0 \leq f \leq g - 3$, let $\nu_{g,f} = f(G_{0,1} + G_{1,0}) + (G_{1,g-f-1} + G_{g-f-1,1})$,
(slopes $0, 1$ with mult. f and $\frac{1}{g-f}, \frac{g-f-1}{g-f}$ with mult. $g - f$.
Note that $\nu_{g,f}$ is the most generic Newton polygon with p-rank f.



$g = 4$, $f = 1$.

Conjecture: let $g \geq 3$ and $0 \leq f \leq g - 3$

The generic point of any component of $\mathcal{M}_g^f$ has Newton polygon $\nu_{g,f}$.

B1 Problem: hyperelliptic $g = 3$, $\text{codim}(\mathcal{H}_3, \mathcal{A}_3) = 1$

Problem: Let p odd and X a hyp. curve of genus $g = 3$.

If X generic p-rank 0, does Cartier matrix on $H^0(X, \Omega^1)$ have rank 2?

$X : y^2 = h(x)$ with $h(x) = x^7 + ax^6 + bx^5 + cx^4 + dx^3 + ex^2 + x$.
Finite-to-1 map $\mathbb{A}^5_k - \Delta \to \mathcal{H}_3$ taking (a, b, c, d, e) to X.

The condition $M_2 M_1 M_0 = [0]$ true for dim 2 subspace in (a, b, c, d, e).
For each component, does M_0 generically have rank 2?

Elkin/P: yes when $p = 3, 5$. If $p = 3$, Cartier operator has matrix

$$M_1 = \begin{bmatrix} e & 1 & 0 \\ b & c & d \\ 0 & 1 & a \end{bmatrix}.$$

If $r \leq 1$, then $e = b = d = a = 0$, and X singular. So $r = 2$ if $f = 0$.

B2 Open question about p-ranks of cyclic covers

Question: Let $\ell \neq p$ odd prime.

Does there exist a $\mathbb{Z}/\ell$-cover $Y \to \mathbb{P}^1$ over $\overline{\mathbb{F}}_p$ such that Y is a smooth curve of genus g and p-rank f?

Not always! There are new constraints on g and f.

Equation: $y^\ell = \prod_{i=1}^n (x - \beta_i)^{a_i}$ where $0 < a_i < \ell$ and $\sum_{i=1}^n a_i \equiv 0 \bmod \ell$.

$a_1, \ldots, a_n$ well-defined up to permutation and sim. mult. by $c \in (\mathbb{Z}/\ell\mathbb{Z})^*$.

Def: Inertia type: $\vec{a} = \{a_1, \ldots, a_n\}$.

Riemann-Hurwitz: $g(Y) = (\ell - 1)(n-2)/2$.

Congruence condition

Let e be the order of p modulo ℓ. Then e divides the p-rank f.

Eigenspaces

Let $\tau \in \mathrm{Aut}(Y)$ with $\tau(y) = \zeta y$ where ζ is primitive ℓth root of unity.

Then τ induces a linear transformation of $H^0(Y, \Omega^1)$.

Decompose $H^0(Y, \Omega^1)$ into eigenspaces: $H^0(Y, \Omega^1) = \oplus_{i=0}^{\ell-1} \mathcal{L}_i$, where $\mathcal{L}_i = \{\omega \in H^0(Y, \Omega^1) \mid \tau^*(\omega) = \zeta^i \omega\}$.

Fact: Let $d_i = \dim(\mathcal{L}_i)$. Then $d_0 = 0$ and $d_i = -1 + \sum_{j=1}^{n} (\frac{ia_j}{\ell} - \lfloor \frac{ia_j}{\ell} \rfloor)$.

Ex: Let $\ell = 3$.
The *signature type* is (r, s) where $r = \dim(\mathcal{L}_1)$ and $s = \dim(\mathcal{L}_1)$.
Note $r + s = g$ and $(g-1)/3 \leq r, s \leq (2g+1)/3$.
There is a bijection between inertia types and signature types:

$$\#\{a_i = 1\} = 2s - r + 1, \ \#\{a_i = 2\} = 2r - s + 1.$$

Upper bound on p-rank

Cartier operator C permutes $\{\mathcal{L}_i \mid 1 \leq i \leq \ell - 1\}$ by $C(\zeta^{pi}\omega) = \zeta^i C(\omega)$ so $C(\mathcal{L}_i) \subset \mathcal{L}_{\sigma(i)}$ where σ is the permutation $i \mapsto p^{-1} i$ mod ℓ of $(\mathbb{Z}/\ell\mathbb{Z})^*$.

Each orbit of $\{\mathcal{L}_i \mid 1 \leq i \leq \ell - 1\}$ under C has length e, where e is the order of p modulo ℓ.

Bouw: (dual result to action of F on $H^1(Y, \mathcal{O})$)

The stable rank of C on $\mathcal{L}_i$ is bounded by $\min\{\dim(\mathcal{L}_i)\}$ across orbit.

Let $B(\vec{a}) = \sum_{\text{orbits } O} e \cdot \min\{\dim(\mathcal{L}_i) \mid i \in O\}$.

Then $f(Y) \leq B(\vec{a})$ for all $\mathbb{Z}/\ell$-covers with inertia type $\vec{a}$.

The upper bound $B(\vec{a})$ occurs as the p-rank for (generic) curve in $\mathcal{T}_{\ell,\vec{a}}$.

Ex: Let $\ell = 3$.
Then $B(\vec{a}) = g$ if $p \equiv 1$ mod 3 and $B(\vec{a}) = 2\min\{r, s\}$ if $p \equiv 2$ mod 3.

An existence result for trielliptic covers

Let $g \geq 3$ and let (r, s) be a trielliptic signature for g.

Suppose either:

1. $p \equiv 2 \bmod 3$ is odd and $0 \leq f \leq 2\min(r, s)$ is even; or
2. $p \equiv 1 \bmod 3$ and $f = g - 2$.

Ozman/P/Weir

Then there exists a $\mathbb{Z}/3$-cover $\phi : Y \to \mathbb{P}^1$ with Y a smooth curve of genus g, trielliptic signature (r, s) and p-rank f.

More generally, $\mathcal{T}^f_{(r,s)}$ is non-empty and contains a component S with $\dim(S) = \max(r, s) - 1 + f/2$ in case (1) and $\dim(S) = f$ in case (2).

B2 Problem about cyclic covers

Let $\ell \neq p$ be prime.

Restrict to $\mathbb{Z}/\ell$-covers of the projective line with 3 branch points. (There are only finitely many of these).

Open question

For $\vec{a} = (a_1, a_2, a_3)$ inertia type for $\mathbb{Z}/\ell$, what is the p-rank of the $\mathbb{Z}/\ell$-cover $Y \to \mathbb{P}^1$ with inertia $\vec{a}$?

Can suppose that $a_2 = 1$.
Reduce to equation $y^\ell = x^{a_1}(x-1)^1 = x^{a_1+1} - x^{a_1}$.

Elkin: formula for Cartier operator on $H^0(X, \Omega^1)$.

Example: trielliptic $g = 4$, signature $(2,2)$

If $g = 4$ and $\dim(L_1) = \dim(L_2) = 2$:

(Note - Torelli locus has codimension 1 in $\mathrm{GU}(2,2)$).

Write $X : y^3 = p_1(x)p_2(x)^2$ where $p_1(x) = x(x^2 - 1)$

and $p_2(x) = x^3 + ax^2 + bx + c$ has distinct roots in $k - \{0, \pm 1\}$.

Basis $\{w_{11} = \frac{dx}{y}, w_{12} = \frac{xdx}{y}\}$ and $\{w_{21} = p_1(x)\frac{dx}{y^2}, w_{22} = p_1(x)\frac{xdx}{y^2}\}$.
Elkin: action of C on basis.

$$C(w_{11}) = f_{1,p-1}(x)w_{21},\ C(w_{12}) = f_{1,p-2}(x)w_{21},$$

$$C(w_{21}) = f_{2,p-1}(x)w_{11},\ C(w_{22}) = f_{2,p-2}(x)w_{11}.$$

The p-rank of X is rank of matrix $M = C^{(p^3)}C^{(p^2)}C^{(p)}C$.

Example: trielliptic $g = 4$ signature $(2,2)$, $p = 5$

Curve $X : y^3 = x(x^2-1)p_2(x)^2$ where $p_2(x) = x^3 + ax^2 + bx + c$.
When $p = 5$, the Cartier matrix C is

$$\begin{bmatrix} 0 & 0 & 4b & 2a+c \\ 0 & 0 & 4c & 2b+3 \\ 4abc+4b^3+3bc^2+2c^2 & a^3+ab+2a+3c & 0 & 0 \\ 2ac^2+2b^2c+c^3 & 3a^2b+2a^2+ac+3b^2+2b & 0 & 0 \end{bmatrix}.$$

$\mathrm{det}(C) = (ab+2c^2+2)^2 \mathrm{disc}(p_2(x))$.

Problem: $M = C^{(125)}C^{(25)}C^{(5)}C$ has 8 non-trivial entries, each a polynomial in 3 variables with 288 monomials, half of degree 208, the other half of degree 416.

Strategy: use resultants, lift solutions.

Example: trielliptic $g = 4$ signature $(2,2)$, $p = 5$

Ozman/P/Weir: Let $p = 5$.

The p-rank 0 strata of the moduli space $T_{2,2}$ of genus 4 trielliptic curves with signature $(2,2)$ has two components, each rational of dimension 1, each intersecting Δ_2.

Proof: Show $V = \{(a,b,c) \in \mathbb{A}^3 : M = 0\}$ has 4 components.

Action of $S_3 = \mathrm{Stab}(0,1,-1)$ permutes 3 of them and fixes one.

The two irreducible components Z and W of $T_{2,2}$ parametrized by $X : y^3 = x(x^2-1)(Dx^3 + Ax^2 + Bx + C)^2$ are

$$\left\{ \begin{array}{l} A = 2u^{10}v + 2u^6v^5 + v^{11} \\ B = u^5v^6 \\ C = u^6v^5 \\ D = 3u^{11} + u^5v^6 + 4uv^{10} \end{array} \right\}, \quad \left\{ \begin{array}{l} A = u^2v + 4v^3 \\ B = uv^2 \\ C = u^2v \\ D = u^3 + 2uv^2 \end{array} \right\}.$$

Review: problems about p-rank

Open question B1: X a generic curve of genus g and p-rank 0

what is the Newton polygon of X?

Guess: NP has slopes $1/g$ and $(g-1)/g$.
equivalently, guess rank of Cartier operator on $H^0(X,\Omega^1)$ is $g-1$.

Test case: generic hyperelliptic curve with $g=3$ and $f=0$.

Open question B2: for $\mathbb{Z}/\ell$-covers $X \to \mathbb{P}^1$

What is p-rank of X?

New conditions on g and f for given ℓ, p.

Big picture: in moduli space $\mathcal{A}_g$, study interaction between Torelli locus, p-rank strata, Hurwitz spaces, and loci of abelian varieties which decompose (with product polarization).

Geometric tools: dimension of p-rank strata

Let p prime and let $\ell \neq p$ be odd prime.
Let e be the order of p modulo ℓ.
Let g be multiple of $(\ell - 1)/2$ and $0 \leq f \leq g$ multiple of e.

Let $T_{\ell,\vec{a}}$ be the Hurwitz space of $\mathbb{Z}/\ell$-covers of $\mathbb{P}^1_k$ with inertia type $\vec{a}$.
Let Γ be a component of the p-rank f strata $T^f_{\ell,\vec{a}}$.

Oort Purity: The Newton polygon can change only in codim 1.

$$\dim(\Gamma) \geq \dim(T_{\ell,\vec{a}}) - (B(\vec{a}) - f)/e.$$

Ex: Let $\ell = 3$ and f even and $p \equiv 2 \bmod 3$. Then

$$\dim(\Gamma) \geq g - 1 - \min(r, s) + f/2.$$

Strategy: prove existence of cyclic covers with given genus and p-rank by proving equality for dimension.

Singular curves of compact type 

Suppose X has two components X_1 and X_2
(of genera g_1 and g_2 and p-ranks f_1 and f_2)

which intersect in exactly one ordinary double point P
(a ramification point under $\mathbb{Z}/\ell$-action on X_1 and on X_2).

[BLR] $J_X \simeq J_{X_1} \times J_{X_2}$.

So $g = g_1 + g_2$ and $f = f_1 + f_2$.

Let Δ_{g_1} be image of clutching morphism:
$\kappa_{g_1,g_2} : \tilde{\mathcal{T}}_{\ell,g_1} \times \tilde{\mathcal{T}}_{\ell,g_2} \to \overline{\mathcal{T}}_{\ell,g_1+g_2}$ where  $\times$  $\mapsto$ .

For $\ell \geq 3$, have admissible condition for inertia types at node.

Singular curves of non-compact type 

Suppose X has two components X_1 and X_2
(of genera g_1 and g_2 and p-ranks f_1 and f_2)

which intersect in ℓ ordinary double points P and Q
(an orbit under $\mathbb{Z}/\ell$-action on X_1 and on X_2).

[BLR] $1 \to (\mathbb{G}_m)^{\ell-1} \to J_X \to J_{X_1} \times J_{X_2} \to 1$.

So $g = g_1 + g_2 + (\ell - 1)$ and $f = f_1 + f_2 + (\ell - 1)$.

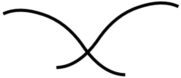
Let Ξ_{g_1} be image of clutching morphism:
$\lambda_{g_1,g_2} : \overline{\mathcal{T}}_{\ell,g_1;1} \times \overline{\mathcal{T}}_{\ell,g_2:1} \to \overline{\mathcal{T}}_{\ell,g_1+g_2+(\ell-1)}$ where .

Singular curves

Suppose X is a singular curve of genus g.

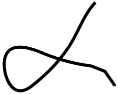
Then X could be reducible.

[Δ_i] For example, it could consist of two irreducible components X_1 of genus i and X_2 of genus $g - i$ intersecting in exactly one ordinary double point.



Or X could be irreducible.

[Δ_0] For example, it could self-intersect in an ordinary double point with normalization an irreducible curve of genus $g - 1$.



Description of 

Suppose X has two components X_1 and X_2
(of genera g_1 and g_2)

which intersect in exactly one ordinary double point P

Then $J_X \simeq J_{X_1} \times J_{X_2}$.

So $g = g_1 + g_2$.

Let Δ_{g_1} be image of clutching morphism:
$\kappa_{g_1,g_2} : \bar{\mathcal{M}}_{g_1;1} \times \bar{\mathcal{M}}_{g_2;1} \to \overline{\mathcal{M}}_{g_1+g_2}$ where .

Description of 

Suppose X self-intersects in one ordinary double point P.

Its normalization X_1 is a curve (of genus g_1).

Then $1 \to \mathbb{G}_m \to J_X \to J_{X_1} \to 1$.

So $g = g_1 + 1$ and $f = f_1 + 1$.

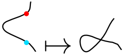
Let Ξ_0 be image of clutching morphism:

$\kappa_g : \overline{\mathcal{M}}_{g;2} \to \overline{\mathcal{M}}_{g+1}$ where  $\mapsto$ .

Geometry of boundary

$\kappa_{g_1,g_2} : \mathcal{M}_{g_1;1} \times \mathcal{M}_{g_2;1} \to \Delta_{g_1}[\mathcal{M}_{g_1+g_2}]$

$\kappa_g : \mathcal{M}_{g-1;2} \to \Delta_0[\mathcal{M}_g]$



Then Δ_i is an irreducible divisor in $\mathcal{M}_g$.

Let $\partial\mathcal{M}_g = \cup_{i=0}^{g/2}\Delta_i$ and $\mathcal{M}_g^0 = \mathcal{M}_g - \partial\mathcal{M}_g$.

Then $\mathcal{M}_g^0$ is the moduli space of *smooth* curves of genus g.

Boundary of $\mathcal{M}_g^f$

Let S be a component of $\mathcal{M}_g^f$.
We prove that S intersects $\partial \mathcal{M}_g$ in every way possible.
Also have similar result about boundary of $\mathcal{H}_g^f$ when $p > 2$.

Theorem (Achter/P)

Let $g_i \in \mathbb{Z}^{\geq 1}$ and $0 \leq f_i \leq g_i$ be such that $\sum g_i = g$ and $\sum f_i = f$. Then S contains a chain of smooth curves Y_i of genus g_i and p-rank f_i.

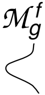
Sketch of proof:
When $f = 0$, follows from result of Faber/van der Geer.

When $f \geq 1$, then $\dim S > 2g - 3$.
So S intersects Δ_0, again by F/vdG.

$1 \to \mathbb{G}_m \to J($  $) \to J($  $) \to 1$

Sketch of proof - continued

If $f \geq 1$, inductive strategy:



Sketch of proof - continued

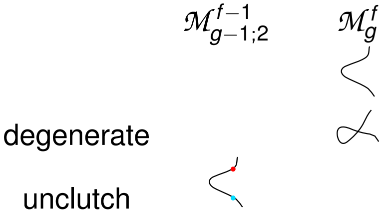
If $f \geq 1$, inductive strategy:

$\mathcal{M}_g^f$

degenerate

Sketch of proof - continued

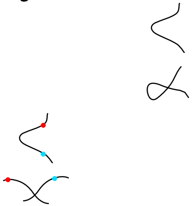
If $f \geq 1$, inductive strategy:



Sketch of proof - continued

If $f \geq 1$, inductive strategy:

$\mathcal{M}_{g-1;2}^{f-1}$ $\mathcal{M}_g^f$



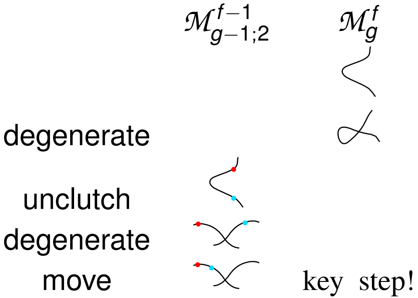
degenerate

unclutch

degenerate

Sketch of proof - continued

If $f \geq 1$, inductive strategy:



Sketch of proof - continued

If $f \geq 1$, inductive strategy: