Open questions on Jacobians of curves over finite fields: supersingular curves

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Effective methods for abelian varieties
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Open question on supersingular curves

Let p be a prime number. Let g be a natural number.

Open question:

Does there exist a supersingular curve of genus g defined over a finite field of characteristic p, for every p and g?

Outline. What is:

1. a supersingular elliptic curve;
2. a supersingular curve of higher genus;
3. known about this question already;
4. the next step?

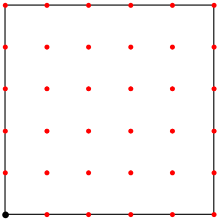
Complex elliptic curves and p-torsion

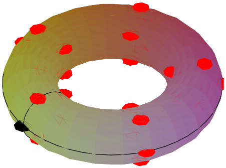
Let E be a complex elliptic curve.

$E \simeq \mathbb{C}/L$ for a lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$.
(Thus E is an abelian group).

Torsion points: $E[p](\mathbb{C}) = \{Q \in E(\mathbb{C}) \mid pQ = 0_E\}$.

Then $E[p](\mathbb{C}) \simeq \frac{1}{p}L/L \simeq (\mathbb{Z}/p)^2$.



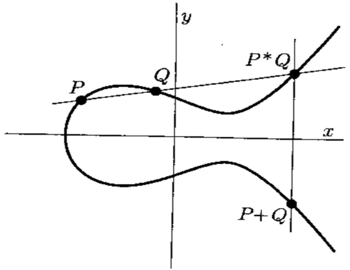


If X is a complex curve of genus $g \geq 2$, its Jacobian J_X is a p.p. abelian variety of dimension g and $J_X[p](\mathbb{C}) \simeq (\mathbb{Z}/p)^{2g}$.

Elliptic curves - algebraic version

Let $E : y^2 = h(x)$ be an elliptic curve over $k = \overline{\mathbb{F}}_p$ where $h(x) = x^3 + ax^2 + bx + c = \prod_{i=1}^{3}(x - \lambda_i)$.

Algebraic group law on E:



The ℓ-torsion of E is $\mathrm{Ker}[\ell]$ where $[\ell] : E \to E$ is mult.by-ℓ.

$E[\ell](k) := \{Q \in E(k) \mid \ell Q = 0_E\} \simeq (\mathbb{Z}/\ell)^2$ if $p \nmid \ell$.

Torsion points - example

Let $E : y^2 = x^3 + ax^2 + bx + c$ and $\ell = 3$.

A point Q has order 3 iff $2Q = -Q$ iff $x(2Q) = x(Q)$.

This occurs iff $x(Q)$ is a root of the 3-division polynomial.

$P. < a, b, c >= PolynomialRing(ZZ, 3)$
$E = EllipticCurve(P, [0, a, 0, b, c])$
$d_3 = E.division\_polynomial(3, x = None)$

$3 * x^4 + 4 * a * x^3 + 6 * b * x^2 + 12 * c * x - b^2 + 4 * a * c$

If $p \neq 3$, then $d_3(x)$ has 4 distinct roots so E has 8 points of order 3 and $|E[3](k)| = 9$.

Collapsing torsion points - example

What if $p = 3$?
$d_3 = 3 * x^4 + 4 * a * x^3 + 6 * b * x^2 + 12 * c * x - b^2 + 4 * a * c$.

Collapsing torsion points - example

What if $p = 3$?
$d_3 = 3*x^4 + 4*a*x^3 + 6*b*x^2 + 12*c*x - b^2 + 4*a*c$.
$P3. < a,b,c >= PolynomialRing(GF(3),3)$

$r_3 = d_3.change\_ring(P3)$

$+a*x^3 - b^2 + a*c$

Mod p binomial thm: In $k[x]$, $(x+\alpha)^p = x^p + \alpha^p$.

So $r_3 = a*x^3 - b^2 + a*c$ has

$$\begin{cases} \text{one (triple) root} & a \not\equiv 0 \bmod 3 \\ \text{no roots} & a \equiv 0 \bmod 3 \end{cases}$$

So $|E[3](k)|$ divides 3 when $p = 3$.

p	r_p reduction of $p-$division polynomial of $y^2 = x^3 + bx + c$
5	$+2 * b * x^{10} - b^2 * c * x^5 + b^6 - 2 * b^3 * c^2 - c^4$
7	$+3 * c * x^{21} + 3 * b^2 * c^2 * x^{14} + (-b^7 * c - 2 * b^4 * c^3 + 3 * b * c^5) * x^7$ $-b^{12} - b^9 * c^2 + 3 * b^6 * c^4 - b^3 * c^6 + 2 * c^8$

Then r_p has at most $(p-1)/2$ roots. The p-torsion points on $E : y^2 = f(x)$ collapse to either p points or 1 point modulo p.

Def:

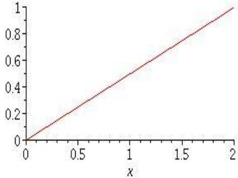
$$E \text{ is } \begin{cases} \text{ordinary} & \text{if } |E[p](k)| = p \\ \text{supersingular} & \text{if } |E[p](k)| = 1 \end{cases}$$

Supersingularity and slopes

If $E/\mathbb{F}_q$ is elliptic curve, then $\#E(\mathbb{F}_q) = q + 1 - a$.
The zeta function of E is $Z(t) = (1 - at + qt^2)/(1-t)(1-qt)$.

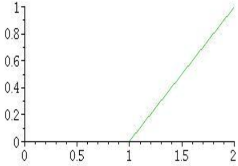
Fact: $p \mid a$ iff E supersingular.

E supersingular, Newton polygon of $1 - at + qt^2$ has slopes $1/2$.



called $G_{1,1}$.

E ordinary, then Newton polygon has slopes 0 and 1.



called $G_{0,1} \oplus G_{1,0}$.

Sage - computing supersingularity

$E = EllipticCurve(GF(5), [0, 1, 0, 2, 0])$
Elliptic Curve defined by $y^2 = x^3 + x^2 + 2 * x$ over Finite Field of size 5
$E.is\_supersingular()$
True
$E.hasse\_invariant()$
0
$E.trace\_of\_frobenius()$
0
$F = E.frobenius()$
$C = F.absolute\_charpoly()$
$x^2 + 5$
$C.newton\_slopes(5)$
$[1/2, 1/2]$

Examples of supersingular elliptic curves

For all p, there exists a supersingular elliptic curve $E/\mathbb{F}_{p^2}$ (Igusa).
The number of isomorphism classes of ss elliptic curves is $\lfloor \frac{p}{12} \rfloor + \varepsilon$.

$p = 2$: $y^2 + y = x^3$ (unique)
$p \equiv 3 \bmod 4$: $y^2 = x^3 - x$
$p \equiv 2 \bmod 3$: $y^2 = x^3 + 1$

p odd: $y^2 = h(x)$, where $h(x)$ cubic with distinct roots, is supersingular iff the coefficient c_{p-1} of x^{p-1} in $h(x)^{(p-1)/2}$ is zero.

This coefficient vanishes iff Cartier operator trivializes $\frac{dx}{y} \in H^0(E, \Omega^1)$.

$$C(\frac{dx}{y}) = C(\frac{y^{p-1}dx}{y^p}) = \frac{1}{y}C(h(x)^{(p-1)/2})dx) = \frac{c_{p-1}^{1/p}dx}{y}.$$

$y^2 = x(x-1)(x-\lambda)$ is supersingular for $\frac{p-1}{2}$ choices of $\lambda \in \overline{\mathbb{F}}_p$ (Igusa).

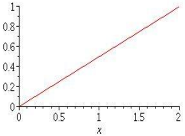
Review: supersingular elliptic curves

Let E be a smooth elliptic curve over $k = \overline{k}$, with $\mathrm{char}(k) = p$.
Let $E[p]$ be the kernel of the inseparable multiplication-by-p morphism.

E is **supersingular** if it satisfies the following equivalent conditions:

A. The only p-torsion point is the identity: $E[p](k) = \{\mathrm{id}\}$.

B. The Newton polygon of E is a line segment of slope $\frac{1}{2}$.



C. The Cartier operator annihilates $H^0(E, \Omega^1)$.

D. $\mathrm{End}(E)$ non-commutative (order in quat. algebra)

Introduction: different properties when $g > 1$

Let A be a p.p. abelian variety of dimension g over $k = \overline{k}$, $\mathrm{char}(k) = p$.
Let $A[p]$ be the kernel of the inseparable multiplication-by-p morphism.

The following conditions are all different for $g \geq 3$.

A. ***p*-rank** 0 - The only p-torsion point is the identity: $A[p](k) = \{\mathrm{id}\}$.

B. supersingular - The Newton polygon of A is a line of slope $\frac{1}{2}$.

C. superspecial - The Cartier operator annihilates $H^0(X, \Omega^1)$.

Then $C \Rightarrow B \Rightarrow A$ but $A \overset{g \geq 3}{\not\Rightarrow} B \overset{g \geq 2}{\not\Rightarrow} C$

Question: if $g \geq 2$, do these occur for Jacobian of smooth k-curve?

Curves of higher genus

Let $k = \overline{\mathbb{F}}_p$ (an algebraically closed field of char. p).

Let X be a (smooth projective connected) curve over k.

Recall: everything you learned about Riemann surfaces ($\mathbb{C}$-curves).

Analogous structures: e.g., functions, differentials, Jacobians.

More complicated definitions: e.g., genus is $g = \dim(H^0(X, \Omega_1))$ rather than 'the number of holes'.

Guideline:

Most facts not involving the number p are still true.

Most facts involving the number p are now false.

Jacobians

Suppose X is a curve. The genus is $g = \dim(H^0(X, \Omega_1))$.

If $g \geq 2$, there is no natural group law on the points of X.

(Recall, define group structure on points of a complex curve by integrating holomorphic differentials and taking quotient by lattice of periods: $J_x = \Omega^1(X)^*/H^1(X, \mathbb{Z}) \simeq \mathbb{C}^g/\mathbb{L}$. Its p-torsion points satisfy $J_X[p](\mathbb{C}) \simeq (\mathbb{Z}/p)^{2g}$.)

Now Jacobian J_X of X is $\mathrm{Pic}^0(X)$ (line bundles of deg 0) or $\mathrm{Div}^0(X)/\mathrm{PDiv(X)}$ (divisors of deg 0 mod principal divisors).

Then J_X is a principally polarized abelian variety of dimension g.

B. Definition of Newton polygon

Let X be a smooth projective curve defined over $\mathbb{F}_q$, with $q = p^a$.
Zeta function of X is $Z(X/\mathbb{F}_q, t) = L(X/\mathbb{F}_q, t)/(1-t)(1-qt)$

where $L(X/\mathbb{F}_q, t) = \prod_{i=1}^{2g}(1 - w_i t) \in \mathbb{Z}[t]$ and $|w_i| = \sqrt{q}$.

The Newton polygon of X is the NP of the L-polynomial $L(t)$.
Find p-adic valuation v_i of coefficient of t^i in $L(t)$.
Draw lower convex hull of $(i, v_i/a)$ where $q = p^a$.

Facts: The NP goes from $(0,0)$ to $(2g, g)$.
NP line segments break at points with integer coefficients;
If slope λ occurs with length m_λ, so does slope $1-\lambda$.

Definition

$X/\mathbb{F}_q$ is *supersingular* if the Newton polygon of $L(X/\mathbb{F}_q, t)$ is a line segment of slope $1/2$.

B. Definition of Newton polygon

Let A be a p.p. abelian variety of dimension g over k.

Manin: for c, d relatively prime s.t. $\lambda = \frac{c}{d} \in \mathbb{Q} \cap [0,1]$,
define a p-divisible group $G_{c,d}$ of dimension c and height d.

The Dieudonné module D_λ for $G_{c,d}$ is a $W(k)$-module.
Over $\mathrm{Frac}(W(k))$, there is a basis $x_1, \dots, x_d$ for D_λ s.t. $F^d x_i = p^c x_i$.

There is an isogeny of p-divisible groups $A[p^\infty] \sim \oplus_\lambda G_{c,d}^{m_\lambda}$.

Newton polygon: lower convex hull - line segments slope λ, length m_λ.

Definition: A *supersingular* iff $\lambda = \frac{1}{2}$ is the only slope.

There is a partial ordering on NPs; the supersingular NP is 'smallest'.

The supersingular property

Let X be a smooth projective curve defined over $\mathbb{F}_q$, with $q = p^a$.
The following are equivalent:

1. X is supersingular;
2. the Newton polygon of $L(X/\mathbb{F}_q, T)$ is a line segment of slope $1/2$;
3. each eigenvalue of the relative Frobenius morphism equals $\zeta\sqrt{q}$ for some root of unity ζ;
4. X is minimal (satisfies lower bound in Hasse-Weil bound for number of points) over $\mathbb{F}_{q^r}$ for some r;
5. Tate: $\mathrm{End}(\mathrm{Jac}(X \times_{\mathbb{F}_q} k)) \otimes \mathbb{Q}_p \simeq M_g(D_p)$, D_p quat alg ram at p, ∞;
6. Oort: $\mathrm{Jac}(X)$ is geometrically isogenous to a product of supersingular elliptic curves.

Motivation for studying supersingular curves

* maximal and minimal curves (supersingular) yield good error-correcting Goppa codes;

* abelian varieties with complex multiplication are often supersingular, useful in cryptography;

* good signature schemes built using supersingular curves;

* supersingular curves play a key role in geometric proofs about stratifications of $\mathcal{A}_g$ by Newton polygon type (or EO type).

Example: Hermitian curves are supersingular

Let $q = p^n$. The *Hermitian curve* X_q has affine equation $y^q + y = x^{q+1}$.

It has genus $g = q(q-1)/2$.

It is maximal over $\mathbb{F}_{q^2}$ because $\# X_q\left(\mathbb{F}_{q^2}\right) = q^3 + 1$.

Ruck/Stichtenoth: X_q is unique curve of genus g maximal over $\mathbb{F}_{q^2}$.

Hansen: X_q is the Deligne-Lusztig variety for $\mathrm{Aut}(X_q) = \mathrm{PGU}(3, q)$.

The L-polynomial of X_q is $L(X_q/\mathbb{F}_q, t) = (1 + qt^2)^g$.

The only slope of the Newton polygon of $L(X_q/\mathbb{F}_q, t)$ is $1/2$.

Thus $\mathrm{Jac}(X_q)$ is supersingular.

Which Newton polygons occur for Jacobians?

For all p and g, there exists:
a supersingular p.p. *abelian variety* of dimension g, namely E^g;
and a supersingular *singular* curve of genus g.

Open question:

Does there exist a supersingular smooth curve of genus g defined over a finite field of characteristic p, for every p and g?

More generally,
which Newton polygons occur for Jacobians of smooth curves?

For $g = 1$ both, $g = 2$ all three, $g = 3$ all five.

Let $\mathcal{A}_g$ be the moduli space of p.p. abelian varieties of dimension g.
The image of $\mathcal{M}_g$ in $\mathcal{A}_g$ is open and dense for $g \leq 3$.

Open question for $g = 4$:

For all p, does there exist a smooth curve of genus 4
which is supersingular? or whose NP has slopes $1/3, 1/2, 2/3$?

$$\begin{tikzcd}
& & \nu_1^1 \oplus \sigma_3 \arrow[dl] & & & & \\
\sigma_4 & \nu_3^0 \oplus \sigma_1 \arrow[l] & & \nu_1^1 \oplus \nu_3^0 \arrow[ul] \arrow[dl] & \nu_4^2 \arrow[l] & \nu_4^3 \arrow[l] & \nu_4^4 \arrow[l] \\
& & \nu_4^0 \arrow[ul] & & & &
\end{tikzcd}$$

$*$: don't know if this NP occurs for Jacobian of smooth curve for all p
$*$: this NP occurs but some components may have problems
$*$: each component has good geometric properties.

(Katz, Oort, Faber/Van der Geer, Pries, Achter-Pries)

Do all NPs occur for Jacobians? Guess - unlikely?

Observation (Oort 2005) $\dim(\mathcal{A}_g) = g(g+1)/2$ and the dimension of the supersingular locus $\mathcal{A}_g[\sigma_g]$ is $\lfloor g^2/4 \rfloor$.

The difference δ_g is length of longest chain of NPs connecting the supersingular NP σ_g to the ordinary NP ν_g.

If $g \geq 9$, then $\delta_g > 3g - 3 = \dim(\mathcal{M}_g)$.

Either (i) $\mathcal{M}_g$ does not admit a perfect stratification by NP (i.e., there are two NPs ξ_1 and ξ_2 such that $\mathcal{A}_g[\xi_1]$ is in the closure of $\mathcal{A}_g[\xi_2]$ but $\mathcal{M}_g[\xi_1]$ is not in the closure of $\mathcal{M}_g[\xi_2]$.)

or (ii) some NPs do not occur for Jacobians of smooth curves.

Test case: $g = 11$ with NP $G_{5,6} \oplus G_{6,5}$ having slopes of $5/11, 6/11$ (does occur when $p = 2$ - Blache).

Do all NPs occur for Jacobians? Evidence?

Only non-existence results are for curves with automorphisms:

Bouw 2001: Not all p-ranks occur for cyclic degree $d > 2$ covers

Especially, not all NPs occur for wildly ramified covers:

Deuring-Shafarevich formula restricts p-rank.

Oort: If $p = 2$, there does not exist a hyp. ss curve of genus 3.

Scholten/Zhu: $p = 2$, $n \geq 2$, there is no hyp. ss curve with $g = 2^n - 1$.

(for odd p, generalized for Artin-Schreier covers $X \stackrel{\mathbb{Z}/p}{\to} \mathbb{P}^1$ by Blache)

But.....

Van der Geer & Van der Vlugt: If $p = 2$, then there exists a supersingular curve of every genus.

Step one of proof by VdG/VdV

Def: $R[x] \in k[x]$ is an additive polynomial if $R(x_1 + x_2) = R(x_1) + R(x_2)$.
Then $R[x] = c_0 x + c_1 x^p + c_2 x^{p^2} + c_h x^{p^h}$.

Supersingular Artin-Schreier curves

If $R[x] \in k[x]$ is an additive polynomial of degree p^h, then
$X : y^p - y = xR[x]$ is supersingular with genus $p^h(p-1)/2$.

Proof: Induction on h, starting with $h = 0$.
Key fact: $\mathrm{Jac}(X)$ is isogenous to a product of Jacobians of Artin-Schreier curves for additive polynomials of smaller degree.

Remark: Bouw et al studied L-polynomials, automorphism groups of X.
Remark: Blache studied first slope of NP of more general AS curves

Existence of supersingular curves when $p = 2$

Van der Geer and Van der Vlugt

If $p = 2$, then there exists a supersingular curve over $\overline{\mathbb{F}}_2$ of every genus.

Proof sketch: Expand g as (with $s_i \leq s_{i-1} + r_{i-1} + 2$)
$g = 2^{s_1}(1 + 2 + \cdots + 2^{r_1}) + 2^{s_2}(1 + 2 + \cdots 2^{r_2}) + \cdots + 2^{s_t}(1 + 2 + \cdots + 2^{r_t})$.

Let $\mathbf{L} = \oplus_{i=1}^{t} L_i$ for L_i subspace of dim $d_i := r_i + 1$ in vector space of additive polynomials of deg 2^{u_i}, with $u_i = (s_i + 1) - \sum_{j=1}^{i-1}(r_j + 1)$.

If $f \in \mathbf{L}$, let $C_f : y^p - y = xf$. Let Y be fiber product of $C_f \to \mathbb{P}^1$ for all $f \in \mathbf{L}$. Then $J_Y \sim \oplus_{f \neq 0} J_{C_f}$ (thus supersingular). Also, $g_Y = \sum_{f \neq 0} g_{C_f}$.

The number of $f \in \mathbf{L}$ which have a non-zero contribution from L_i, but not from L_j for $j > i$, is $(2^{d_i} - 1)\prod_{j=1}^{i-1} 2^{d_j}$. Each adds $2^{u_i - 1}$ to g.
So $g_Y = \sum_{i=1}^{t}(2^{d_i} - 1)\prod_{j=1}^{i-1} 2^{d_j} 2^{u_i - 1} = \sum_{i=1}^{t} 2^{s_i}(1 + \cdots + 2^{r_i}) = g$.

Supersingular Artin-Schreier curves for odd p

Here is what VdG/VdV's method produces for odd p.

Karemaker/P

Let $g = Gp(p-1)^2/2$ where $G = \sum_{i=1}^{t} p^{s_i}(1+p+\cdots p^{r_i})$. Then there exists a supersingular curve over $\overline{\mathbb{F}}_p$ of genus g.

Can this be improved?

VdG/VdV also prove that there exists a supersingular curve defined over $\mathbb{F}_2$ of every genus. The construction is a little more complicated.

Related question: the p-rank of X

If X is a smooth k-curve of genus g,

Fact/Def:

then $|J_X[p](k)| = p^f$ for some integer $0 \leq f \leq g$ called the p-rank of X.

Also, $f = \dim_{\mathbb{F}_p} \mathrm{Hom}(\mu_p, J_X[p])$ where
$\mu_p \simeq \mathrm{Spec}(k[x]/(x^p - 1))$ is the kernel of Frobenius on $\mathbb{G}_m$.

Let $L(t)$ be the L-polynomial of the zeta function of an $\mathbb{F}_q$-curve X.

The p-rank of X is the length of the slope 0 portion of $\mathrm{NP}(X)$.

X is supersingular if all slopes of $\mathrm{NP}(X)$ equal $1/2$.
X supersingular implies X has p-rank 0 but converse false for $g \geq 3$.

Existence of curves with given genus and p-rank

Let $g \in \mathbb{N}$, $0 \leq f \leq g$ and p prime.

The moduli space $\mathcal{M}_g$ (resp. $\mathcal{H}_g$) of (hyperelliptic) curves of genus g can be stratified by p-rank into strata $\mathcal{M}_g^f$ (resp. $\mathcal{H}_g^f$) whose points represent (hyperelliptic) curves of genus g and p-rank f.

Theorem: Faber/Van der Geer

Every component of $\mathcal{M}_g^f$ has dimension $2g - 3 + f$;
there exists a smooth curve over $\overline{\mathbb{F}}_p$ with genus g and p-rank f.

Theorem: Glass/P (p odd), P/Zhu (p even)

Every component of $\mathcal{H}_g^f$ has dimension $g - 1 + f$;
there exists a smooth hyp. curve over $\overline{\mathbb{F}}_p$ with genus g and p-rank f.

In most cases, it is not known whether $\mathcal{M}_g^f$ and $\mathcal{H}_g^f$ are irreducible.

Supersingular versus p-rank 0

Let A/k be a p.p. abelian variety of dimension g.

Fact: If A is supersingular, then A has p-rank 0.

If $g \in \{1,2\}$ and A has p-rank 0, then A is supersingular.
If $g \geq 3$ and A has p-rank 0, then A usually not supersingular.

Example: Let $j \in \mathbb{N}$ with $p \nmid j$ and $h(x) \in k[x]$ of degree j.
The curve $X : y^q + y = h(x)$ has genus $g = (q-1)(j-1)/2$.
Deuring-Shafarevich formula: $\mathrm{Jac}(X)$ has p-rank 0.

Zhu: Let $q = 2$ and $j = 2^{n+1} - 1$, none of the 2-rank 0 curves $y^2 + y = h(x)$ are supersingular.

Moduli of curves: supersingular versus p-rank 0

Oort: There exists a hyperelliptic curve of genus 3 with p-rank 0 which is not supersingular.

proof: study intersection of two codim 1 conditions in $\mathcal{M}_3^0$.

Application - Achter/P. Let $g \geq 3$ and $p \neq 2$ for hyperelliptic

The generic point of any component of the p-rank 0 strata $\mathcal{M}_g^0$ and $\mathcal{H}_g^0$ is not supersingular.

$A \not\Rightarrow B$ for curves:
if $g \geq 3$, there exists a (hyperelliptic) curve of genus g with p-rank 0 which is not supersingular.

Newton polygon results for $f = g - 3$ and $f = g - 4$

For $g \geq 4$ and $g - 2 \leq f \leq g$, the p-rank determines the Newton polygon (and so that Newton polygon occurs, open and dense in $\mathcal{M}_g^f$.

Let $\nu_{g,f} = f(G_{0,1} + G_{1,0}) + (G_{1,g-f-1} + G_{g-f-1,1})$.

Application - Achter/P. Let $g \geq 3$ and $f = g - 3$.

The generic point of each component of $\mathcal{M}_g^{g-3}$ has Newton polygon $\nu_{g,g-3}$ (slopes $0, \frac{1}{3}, \frac{2}{3}, 1$).

Application - Achter/P. Let $g \geq 4$ and $f = g - 4$.

The generic point of *at least one* component of $\mathcal{M}_g^f$ has Newton polygon $\nu_{g,g-4}$ (slopes $0, \frac{1}{4}, \frac{3}{4}, 1$).

Note: When $g = 4$, there is *at most one* component of $\mathcal{M}_4^0$ whose generic NP is not $\nu_{4,0}$. If so, the NP has slopes $\frac{1}{3}, \frac{1}{2}, \frac{2}{3}$.

Proof: inductive strategy, reduce to p-rank $f = 0$

Let ν_r be a NP type with p-rank 0 occurring in dimension r.

Let $c_r = \mathrm{codim}(\mathcal{A}_g[\nu_r], \mathcal{A}_g)$.

For $g \geq r$, let ν_g be the NP type with p-rank $g - r$ 'containing' ν_r

$(\nu_g = (G_{0,1} \oplus G_{1,0})^{g-r} \oplus \nu_r)$, add $g - r$ slopes of $0, 1$.

Proposition P

If there exists a component S_r of $\mathcal{M}_r[\nu_r]$ s.t. $\mathrm{codim}(S_r, \mathcal{M}_r) = c_r$,

then, for all $g \geq r$,

there exists a component S_g of $\mathcal{M}_g[\nu_g]$ s.t. $\mathrm{codim}(S_g, \mathcal{M}_g) = c_r$.