p-adic heights on Jacobians of hyperelliptic curves I

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p-adic heights

Let

- K be a number field
- J/K be the Jacobian of a smooth projective curve X (e.g., an elliptic curve)
- p be a prime of good reduction for X and ordinary reduction for J.

In these lectures, we'll discuss global p-adic height pairings

$$h : J(K) \times J(K) \to \mathbf{Q}_p.$$

- While there are many parallels with the theory of the canonical height (presented in Müller's lectures), one key difference is that there may be *many* canonical p-adic valued pairings! (More later.)

Canonical and p-adic heights: differences and similarities

Here are a few interesting differences and similarities between canonical and p-adic heights:

- When $K = \mathbf{Q}$, there is just one p-adic height (up to nontrivial scalar multiple), the *cyclotomic p-adic height*. To make our lives easier, we will spend most of our time working over $K = \mathbf{Q}$.
- h is a bilinear pairing. It is symmetric iff certain (very reasonable) choices are made. (More later.) When we need to, let's go ahead and make these reasonable choices.
- For P torsion, $h(P) = 0$. Does $h(P) = 0$ imply P torsion? Not necessarily. Also, nondegeneracy of the cyclotomic p-adic height for elliptic curves over $\mathbf{Q}$ is already rather mysterious. (More later.)

There are quite a few things that are different in the p-adic world; nevertheless p-adic heights are also useful for explicit methods. We will highlight several applications.

Outline

- Motivation
- Cyclotomic p-adic height on elliptic curves over $\mathbf{Q}$
- Anticyclotomic p-adic height on elliptic curves over quadratic imaginary number fields

Motivation

Why compute p-adic heights?

p-adic Birch and Swinnerton-Dyer conjecture

- Mazur-Tate-Teitelbaum ’86: stated the conjecture for elliptic curves and gave numerical evidence
- Mazur-Tate ’91: p-adic heights in terms of p-adic sigma function
- Wuthrich ’04: variation of p-adic height in a family of elliptic curves
- Mazur-Stein-Tate ’06 (and Harvey ’08): fast method for computing cyclotomic p-adic height for elliptic curves
- Stein-Wuthrich ’13: fast method for computing p-primary part of Shafarevich-Tate group for elliptic curves when p-descents are impractical and also where no other methods are known (e.g., Mordell-Weil rank at least 2)
- B.-Müller-Stein ’15: stated conjecture for modular abelian varieties, with data for modular abelian surfaces

Why compute p-adic heights?

Examples of Kim's nonabelian Chabauty method to find integral or rational points on curves, in the spirit of explicit Mordell

- Kim, B.-Kedlaya-Kim '10: integral points on elliptic curves of rank 1
- B.-Besser-Müller '13: integral points on genus g hyperelliptic curves whose Jacobians have Mordell-Weil rank g
- B.-Dogra '16: rational points on genus 2 bielliptic curves whose Jacobians have Mordell-Weil rank 2

p-adic heights on elliptic curves

Let p be an odd prime and let E be an elliptic curve over a number field K with good ordinary reduction at p.

- A p-adic height pairing is a symmetric bilinear pairing

$$(\quad,\quad):E(K)\times E(K)\to\mathbf{Q}_p.$$

- p-adic height pairings were
  - First defined for abelian varieties by Schneider ('82), Mazur-Tate ('83),
  - extended to motives by Nekovář ('93),
  - also defined, in the case of Jacobians of curves, by Coleman and Gross ('89).
  - This third definition is known to be equivalent to the previous ones (Besser, '04).

Birch and Swinnerton-Dyer conjecture

Conjecture (Birch–Swinnerton-Dyer)

Let E be an elliptic curve over $\mathbf{Q}$. Then we have

$$r := \mathrm{rk}(E/\mathbf{Q}) = \mathrm{ord}_{s=1} L(E,s)$$

and

$$L^*(E,1) = \frac{\mathrm{Reg}(E/\mathbf{Q}) \cdot \Omega \cdot |Ш(E/\mathbf{Q})| \cdot \prod_v c_v(E)}{|E(\mathbf{Q})_{tors}|^2},$$

where $L^*(E,1)$ is the leading coefficient of $L(E,s)$ and $\mathrm{Reg}(E/\mathbf{Q})$ is the regulator, defined using the real-valued Néron-Tate height pairing.

p-adic Birch and Swinnerton-Dyer conjecture

Conjecture (Mazur–Tate–Teitelbaum)

Let E be an elliptic curve over $\mathbf{Q}$ with good, ordinary reduction at p. Then we have

$$r := \mathrm{rk}(E/\mathbf{Q}) = \mathrm{ord}_{T=0}(\mathcal{L}_p(E,T))$$

and

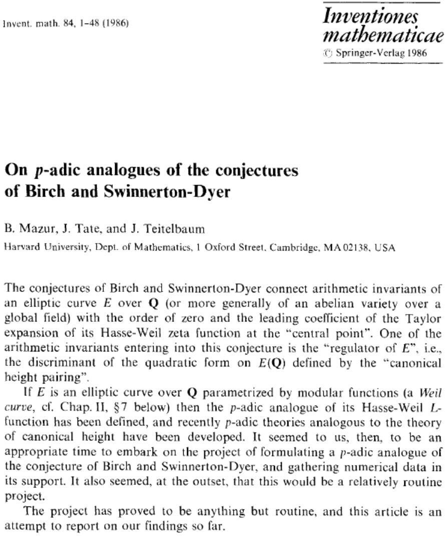
$$\mathcal{L}_p^*(E,0) = \epsilon_p \frac{\mathrm{Reg}_\gamma(E/\mathbf{Q}) \cdot |Ш(E/\mathbf{Q})| \cdot \prod_v c_v(E)}{|E(\mathbf{Q})_{tors}|^2},$$

where $\mathcal{L}_p^*(E,0)$ is the leading coefficient of the p-adic L-function $\mathcal{L}_p(E,T)$ and

$$\mathrm{Reg}_\gamma(E/\mathbf{Q}) = \mathrm{Reg}_p(E/\mathbf{Q})/\log_p(\gamma)^r,$$

with $\mathrm{Reg}_p(E/\mathbf{Q})$ the p-adic regulator, defined using the cyclotomic p-adic height pairing, a p-adic analogue of the real-valued Néron-Tate height pairing.

Data in Mazur-Tate-Teitelbaum



Invent. math. 84, 1–48 (1986)

Inventiones mathematicae
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On p-adic analogues of the conjectures of Birch and Swinnerton-Dyer

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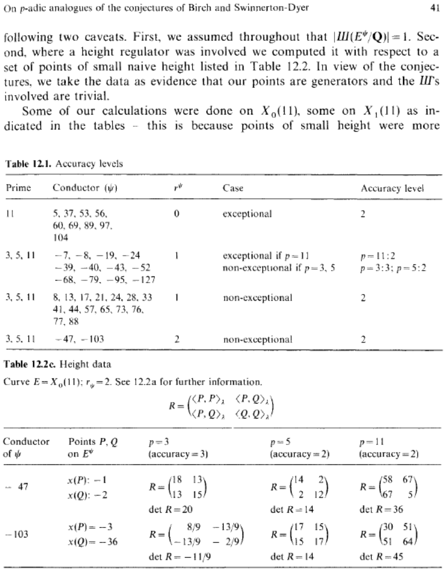
The conjectures of Birch and Swinnerton-Dyer connect arithmetic invariants of an elliptic curve E over **Q** (or more generally of an abelian variety over a global field) with the order of zero and the leading coefficient of the Taylor expansion of its Hasse-Weil zeta function at the "central point". One of the arithmetic invariants entering into this conjecture is the "regulator of E", i.e., the discriminant of the quadratic form on $E(\mathbf{Q})$ defined by the "canonical height pairing".

If E is an elliptic curve over **Q** parametrized by modular functions (a *Weil curve*, cf. Chap. II, §7 below) then the p-adic analogue of its Hasse-Weil L-function has been defined, and recently p-adic theories analogous to the theory of canonical height have been developed. It seemed to us, then, to be an appropriate time to embark on the project of formulating a p-adic analogue of the conjecture of Birch and Swinnerton-Dyer, and gathering numerical data in its support. It also seemed, at the outset, that this would be a relatively routine project.

The project has proved to be anything but routine, and this article is an attempt to report on our findings so far.



following two caveats. First, we assumed throughout that $|Ш(E^{\psi}/\mathbf{Q})|=1$. Second, where a height regulator was involved we computed it with respect to a set of points of small naive height listed in Table 12.2. In view of the conjectures, we take the data as evidence that our points are generators and the Ш's involved are trivial.

Some of our calculations were done on $X_0(11)$, some on $X_1(11)$ as indicated in the tables – this is because points of small height were more

Table 12.1. Accuracy levels

Prime	Conductor (ψ)	r^{ψ}	Case	Accuracy level
11	5, 37, 53, 56, 60, 69, 89, 97, 104	0	exceptional	2
3, 5, 11	−7, −8, −19, −24 −39, −40, −43, −52 −68, −79, −95, −127	1	exceptional if $p=11$ non-exceptional if $p=3, 5$	$p=11:2$ $p=3:3$; $p=5:2$
3, 5, 11	8, 13, 17, 21, 24, 28, 33 41, 44, 57, 65, 73, 76, 77, 88	1	non-exceptional	2
3, 5, 11	−47, −103	2	non-exceptional	2

Table 12.2c. Height data

Curve $E=X_0(11)$; $r_{\psi}=2$. See 12.2a for further information.

$$R=\begin{pmatrix}\langle P,P\rangle_{\lambda} & \langle P,Q\rangle_{\lambda}\\ \langle P,Q\rangle_{\lambda} & \langle Q,Q\rangle_{\lambda}\end{pmatrix}$$

Conductor of ψ	Points P, Q on E^{ψ}	$p=3$ (accuracy = 3)	$p=5$ (accuracy = 2)	$p=11$ (accuracy = 2)
−47	$x(P)$: −1 $x(Q)$: −2	$R=\begin{pmatrix}18 & 13\\ 13 & 15\end{pmatrix}$ $\det R=20$	$R=\begin{pmatrix}14 & 2\\ 2 & 12\end{pmatrix}$ $\det R=14$	$R=\begin{pmatrix}58 & 67\\ 67 & 5\end{pmatrix}$ $\det R=36$
−103	$x(P)=-3$ $x(Q)=-36$	$R=\begin{pmatrix}8/9 & -13/9\\ -13/9 & -2/9\end{pmatrix}$ $\det R=-11/9$	$R=\begin{pmatrix}17 & 15\\ 15 & 17\end{pmatrix}$ $\det R=14$	$R=\begin{pmatrix}30 & 51\\ 51 & 64\end{pmatrix}$ $\det R=45$

p-adic heights: p-adic BSD and Ш

In fact, more is known about p-adic BSD than classical BSD.

- By work of Kato, the computation of an approximation of the p-adic L-series of E for an odd prime p of good reduction produces an *upper bound* on the rank r of the Mordell-Weil group $E(\mathbf{Q})$!

Moreover, explicitly computing p-adic heights and regulators plays an important role in the following:

Theorem (Stein-Wuthrich)

Let $E/\mathbf{Q}$ be the rank 2 elliptic curve 389a1. Then for 2 and all 5005 good ordinary primes $p < 48859$ except $p = 16231$ we have

$$Ш(E/\mathbf{Q})[p] = 0.$$

p-adic heights and K-rational points

Theorem (B.-Dogra-Müller ’16)

Consider $X_0(37)$ *with affine model*

$$y^2 = -x^6 - 9x^4 - 11x^2 + 37.$$

Then $X_0(37)(\mathbf{Q}(i)) = \{(\pm 2i, \pm 1), (\pm 1, \pm 4), \infty^{\pm}\}$.

Remarks:

- The proof of this result involves
  - studying relationships between p-adic heights on elliptic curves over number fields, as well as
  - explicit computation of p-adic heights!
- Note that we have that $\operatorname{rk} J_0(37)(\mathbf{Q}(i)) = 2$, so this is not amenable to the Chabauty-Coleman method.

Cyclotomic p-adic height on $E/\mathbf{Q}$

Computing cyclotomic p-adic height on $E/\mathbf{Q}$

Let

- E be an elliptic curve over $\mathbf{Q}$,
- p a good, ordinary prime for E.

In this scenario, there is (up to scalar multiple) only one p-adic height, the *cyclotomic p-adic height*.

Suppose $P \in E(\mathbf{Q})$ is a non-torsion point

- that reduces to $\mathcal{O} \in E(\mathbf{F}_p)$
- and to a nonsingular point of $E(\mathbf{F}_\ell)$ for all primes ℓ at which E has bad reduction.

Mazur-Stein-Tate ('06) gave a fast way to compute the cyclotomic p-adic height h:

$$h(P) = \frac{1}{p} \log_p \left(\frac{\sigma_p(P)}{d(P)} \right).$$

$\sigma_p(P), d(P)$

Suppose E is given by a model $y^2 = x^3 + Ax + B$, with $A, B \in \mathbf{Z}$. We define the p-adic sigma function and the denominator function:

- p-adic σ function σ_p: the unique odd function $\sigma_p(t) = t + \cdots \in t\mathbf{Z}_p[[t]]$ satisfying

  $$x(t) + c = -\frac{d}{\omega}\left(\frac{1}{\sigma_p}\frac{d\sigma_p}{\omega}\right)$$

  (with ω the invariant differential $\frac{dx}{2y}$ and $c \in \mathbf{Z}_p$, which can be computed by Kedlaya's algorithm)
- denominator function $d(P)$: if $P = (x_P, y_P) = \left(\frac{a_P}{d_P^2}, \frac{b_P}{d_P^3}\right)$, then $d(P) = d_P$

The height pairing

We use $h(nP) = n^2 h(P)$ to extend the height to the full Mordell-Weil group.

The symmetric, bilinear pairing is defined by

$$\begin{aligned} E(\mathbf{Q}) \times E(\mathbf{Q}) &\to \mathbf{Q}_p \\ (\quad P \quad , \quad Q \quad) &\mapsto h(P) + h(Q) - h(P+Q) \end{aligned}$$

Example: computing a cyclotomic p-adic height

Let E be the rank 1 curve $y^2 + y = x^3 - x$ of conductor 37. The point $P = (0,0)$ is a generator for $E(\mathbf{Q})$. We compute the p-adic height of P for the good ordinary prime $p = 5$.

- The component group of $\mathcal{E}_{\mathbf{F}_{37}}$ is trivial. The reduction of P to $E(\mathbf{F}_5)$ has order 8, so we let
$$Q = 8P = \left(\frac{21}{25}, -\frac{69}{125}\right).$$
We will compute $h(Q) = h(8P)$ and then use $h(P) = \frac{1}{64}h(8P)$.
- Denominator: We have $d(P) = 5$.
- σ_5: solve the differential equation defining the 5-adic sigma function σ_5:
$$x(t) + c = -\frac{d}{\omega}\left(\frac{1}{\sigma_5}\frac{d\sigma_5}{\omega}\right),$$

Computing a p-adic height

- This gives
$$\sigma_5(t) = t + \frac{1}{2}ct^3 + \frac{1}{2}t^4 + \left(\frac{1}{8}c^2 - \frac{5}{12}\right)t^5 + \frac{3}{4}ct^6 + \cdots,$$
where
$$c = \frac{1}{12}\mathbf{E}_2(E,\omega) = 1 + 5 + 4\cdot 5^2 + 5^3 + 5^4 + 5^6 + 4\cdot 5^7 + O(5^8).$$
- Recall $Q = 8P = \left(\frac{21}{25}, -\frac{69}{125}\right)$. So $t = -\frac{x(Q)}{y(Q)} = \frac{35}{23}$ and $\sigma_5(t) = 4\cdot 5 + 5^2 + 5^3 + 5^4 + 2\cdot 5^6 + 3\cdot 5^8 + O(5^9)$.
- So
$$\begin{aligned} h(Q) &= \frac{1}{5}\log_5\left(\frac{4\cdot 5 + 5^2 + 5^3 + 5^4 + 2\cdot 5^6 + 3\cdot 5^8 + O(5^9)}{5}\right) \\ &= 3 + 5 + 2\cdot 5^3 + 3\cdot 5^4 + 3\cdot 5^5 + 2\cdot 5^6 + O(5^8), \end{aligned}$$
- Finally,
$$h(P) = \frac{1}{64}h(Q) = 2+4\cdot5+5^2+2\cdot5^3+2\cdot5^4+3\cdot5^5+2\cdot5^6+O(5^7).$$

p-adic heights on elliptic curves over quadratic imaginary number fields

From **Q** to more general number fields

Unlike the $\mathbf{R}$-valued canonical height, there may be *many* canonical p-adic valued heights associated to E/K for a given number field K.

- Up to nontrivial scalar multiple:

$$\{\text{canonical}\, p\text{-adic height pairings}\} \overset{1:1}{\longleftrightarrow} \{\mathbf{Z}_p\text{-extensions}\, L/K\}$$

- Next interesting case is K quadratic imaginary: here we have two $\mathbf{Z}_p$ extensions, and we study cyclotomic and *anticyclotomic* p-adic heights

Anticyclotomic p-adic height on $E(K)$

- Setup:
  - K a quadratic imaginary number field
  - $p = \pi\pi^c$ a prime split in K
  - E/K has good ordinary reduction at the primes above p.
- Suppose $P \in E(K)$ is a non-torsion point that reduces to
  - 0 in $E(\mathbf{F}_\pi)$ and $E(\mathbf{F}_{\pi^c})$ and to
  - the connected component of all special fibers of the Néron model of E
- The anticyclotomic p-adic height $h^{\text{anti}} := h_\rho$ is given by

$$h^{\text{anti}}(P) = \rho_\pi(\sigma_\pi(P)) - \rho_\pi(\sigma_\pi(P^c)) + \sum_{w \nmid p} \rho_w(d_w(P)),$$

where ρ is the anticyclotomic idele class character ($\rho \circ c = -\rho$ for c complex conjugation).

A key difference between cyclotomic and anticyclotomic

Conjecture (Schneider)

The cyclotomic height pairing is nondegenerate; equivalently the associated p-adic regulator is nonzero.

- However, other p-adic height pairings need not be nondegenerate!
- For $E/\mathbf{Q}$ with good ordinary reduction at p and K quadratic imaginary over which $E(K)$ has odd rank, the anticyclotomic p-adic height pairing for E/K is *not nondegenerate*!

Anticyclotomic heights: computational issues

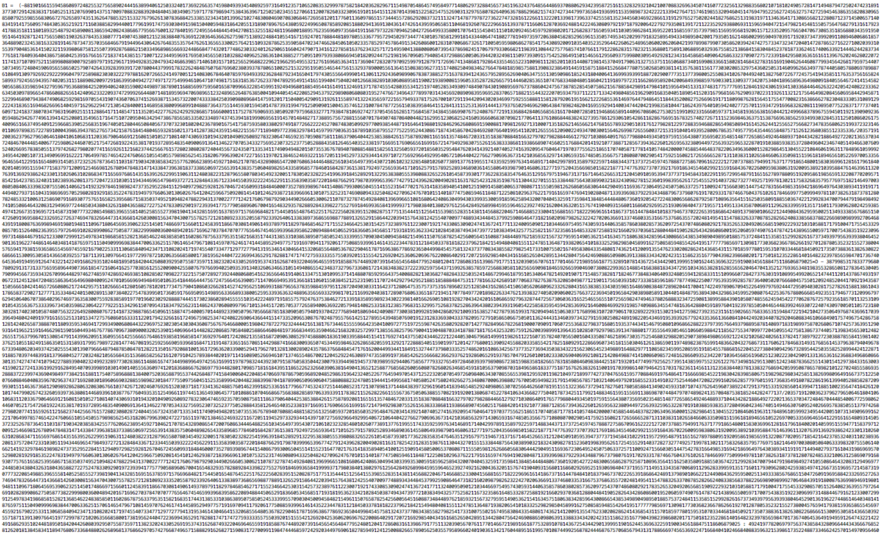
The anticyclotomic p-adic height can be expressed as

$$h^{\text{anti}}(P) = \rho_\pi \left(\frac{\sigma_\pi(P)}{\sigma_\pi(P^c)} \right) + \sum_{\substack{\ell = \lambda\lambda^c \\ \ell \neq p}} \rho_\lambda \left(\frac{d_\lambda(P)}{d_{\lambda^c}(P)^c} \right).$$

Computing the anticyclotomic p-adic height poses two new challenges:

- We begin by computing n such that nP and nP^c to reduce to $0 \in E(\mathbf{F}_p)$. How do we deal with the (typically, very large) multiple of P that results? In particular:
- How do we determine the finite set of split primes which contribute to said point's denominator?

Example (yikes!)



Anticyclotomic height: some packaging

Main challenge: contributions from primes not dividing p.

- Consider the ideal $(x_P)\mathcal{O}_K$ and let $\delta(P) \subset \mathcal{O}_K$ be its denominator ideal.
- Fix $\mathbf{d}_h(P) \in \mathcal{O}_K$ as follows:

$$\mathbf{d}_h(P)\mathcal{O}_K = \prod_{\mathfrak{q}} \mathfrak{q}^{h\,\mathrm{ord}_{\mathfrak{q}}(\delta(P))/2}$$

  where h is the class number of K, and the product is over all prime ideals $\mathfrak{q}$ in $\mathcal{O}_K$.
- Fix an identification $\psi : K_\pi \simeq \mathbf{Q}_p$. We have:

Proposition

The anticyclotomic p-adic height of $P \in E(K)$ *is*

$$h_\rho(P) = \frac{1}{p}\log_p\left(\psi\left(\frac{\sigma_\pi(P)}{\sigma_\pi(P^c)}\right)\right) + \frac{1}{hp}\log_p\left(\psi\left(\frac{\mathbf{d}_h(P)^c}{\mathbf{d}_h(P)}\right)\right).$$

Example

Let E be the elliptic curve "389.a1" given by

$$y^2 + y = x^3 + x^2 - 2x.$$

- analytic rank of $E/\mathbf{Q}$ is 2; algebraic rank of $E(\mathbf{Q})$ is 2
- Let $K = \mathbf{Q}(\sqrt{-11})$; we see $E(K)$ has rank 3.
- 5 is a good ordinary split prime in K.
- Consider $A_1 = \left(-\frac{6}{25}\sqrt{-11} + \frac{27}{25}, -\frac{62}{125}\sqrt{-11} + \frac{29}{125}\right) \in E(K)$. We compute $h^{\text{anti}}(A_1)$.
- Let $(5) = \pi\pi^c$ in $\mathcal{O}_K$, where $\pi = (\frac{1}{2}\sqrt{-11} + \frac{3}{2})$. This allows us to fix an identification

  $$\psi : K_\pi \to \mathbf{Q}_5$$

  that sends

  $$\frac{1}{2}\sqrt{-11}+\frac{3}{2} \mapsto 2{\cdot}5+5^2+3{\cdot}5^3+4{\cdot}5^4+4{\cdot}5^5+3{\cdot}5^7+5^8+5^9+O(5^{10}).$$

Example, continued

- Note the Tamagawa number at 389 is trivial, i.e., $c_{389} = 1$; $n = 9$ is the smallest multiple of A_1 and A_1^c such that both points reduce to 0 in $E(\mathbf{F}_\pi)$. Set $T = 9A_1$.
- Note that the class number of K is $h = 1$. We find $\mathbf{d}_h(A_1) = \frac{1}{2}\sqrt{-11} - \frac{3}{2}$.
- Let f_9 denote the 9th division polynomial associated to E. We compute

$$\begin{aligned}\mathbf{d}_h(T) &= \mathbf{d}_h(9A_1)\\ &= f_9(A_1)\mathbf{d}_h(A_1)^{9^2}\\ &= 24227041862247516754088925710922259344570\sqrt{-11}\\ &\quad - 147355399895912034115896942557395263175125\end{aligned}$$

- We compute

$$\begin{aligned}\sigma_\pi(t) &:= \sigma_5(t)\\ &= t + \left(4 + 5 + 3\cdot 5^2 + 5^3 + 2\cdot 5^4 + 3\cdot 5^5 + O(5^6)\right)t^3 + \cdots\end{aligned}$$

Example, continued

- We compute

$$
\begin{aligned}
h^{\mathrm{anti}}(T) &= \frac{1}{5}\log_5\left(\psi\left(\frac{\sigma_\pi(T)}{\sigma_\pi(T^c)}\right)\right) + \frac{1}{5\cdot 1}\log_5\left(\psi\left(\frac{\mathbf{d}_h(T)^c}{\mathbf{d}_h(T)}\right)\right)\\
&= \frac{1}{5}\log_5\left(\frac{\sigma_5\left(\psi\left(\frac{-x(T)}{y(T)}\right)\right)}{\sigma_5\left(\psi\left(\frac{-x(T)^c}{y(T)^c}\right)\right)}\right) + \frac{1}{5}\log_5\left(\psi\left(\frac{\mathbf{d}_h(T)^c}{\mathbf{d}_h(T)}\right)\right)\\
&= 3 + 5 + 5^2 + 4\cdot 5^4 + 3\cdot 5^5 + 4\cdot 5^7 + 3\cdot 5^8 + 5^9 + O(5^{10})
\end{aligned}
$$

- From this, we obtain the anticyclotomic 5-adic height of A_1:

$$
\begin{aligned}
h^{\mathrm{anti}}(A_1) &= \frac{1}{9^2}h^{\mathrm{anti}}(T)\\
&= 3 + 3\cdot 5 + 3\cdot 5^2 + 2\cdot 5^4 + 4\cdot 5^5 + 4\cdot 5^6 + O(5^8).
\end{aligned}
$$

The p-adic sigma function

Note the important role played by the p-adic sigma function in the definition of these p-adic heights. Recall σ_p satisfies

$$x(t) + c = -\frac{d}{\omega}\left(\frac{1}{\sigma_p}\frac{d\sigma_p}{\omega}\right), \quad \omega = \frac{dx}{2y}.$$

What if we were to try to solve this (p-adic) differential equation?

$$\begin{aligned}
x\frac{dx}{2y} + c\frac{dx}{2y} &= -d\left(\frac{1}{\sigma_p}\frac{d\sigma_p}{\omega}\right)\\
\int\left(x\frac{dx}{2y} + c\frac{dx}{2y}\right) &= -\left(\frac{1}{\sigma_p}\frac{d\sigma_p}{\omega}\right)\\
\frac{dx}{2y}\left(\int\left(x\frac{dx}{2y} + c\frac{dx}{2y}\right)\right) &= -d\log(\sigma_p)\\
\int\frac{dx}{2y}\left(\int\left(x\frac{dx}{2y} + c\frac{dx}{2y}\right)\right) &= -\log(\sigma_p)
\end{aligned}$$

Nonabelian Chabauty

Our second look at p-adic heights is motivated by Kim's nonabelian Chabauty program:

Theorem (Kim, B.-Kedlaya-Kim, '10.)

Let $E/\mathbf{Q}$ be an elliptic curve with rank 1 such that the given model is minimal and all Tamagawa numbers are 1. Then the ratio given by Coleman integrals

$$\frac{\int_b^P \frac{dx}{2y}\,\frac{xdx}{2y}}{\left(\int_b^P \frac{dx}{2y}\right)^2},$$

is constant on non-torsion integral points P.

Reinterpreted: nonabelian Chabauty, $g = r = 1$ at "level 2"

The previous theorem could be thought of as giving us a function which helps cut out integral points:

- For the elliptic curve $y^2 = x^3 + ax + b$, (with rank 1 and squarefree discriminant), consider

$$\log(z) := \int_b^z \frac{dx}{2y}, \qquad D_2(z) = \int_b^z \frac{dx}{2y}\frac{xdx}{2y}.$$

- By writing $\log(z)$ and $D_2(z)$ as p-adic power series and fixing one integral point P, one can consider

$$H(z) := D_2(z)\log^2(P) - D_2(P)\log^2(z).$$

- B-Kedlaya Kim: integral points on an elliptic curve are contained in the set of zeros of $\{z : H(z) = 0\}$.

How do we extend this to higher genus curves?